SOME ESTIMATES OF UNIVALENT AND MULTIVALENT ANALYTIC FUNCTIONS

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

BY
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M. Sc.

to the

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY KANPUR
JULY 1970

DEDICATED TO THE MEMORY OF MY PARENTS

CERTIFICATE

This is to certify that the work embodied in the thesis "Some Estimates of Univalent and Multivalent Analytic Functions" by S.K. Bajpai has been carried out under my supervision and has not been submitted elsewhere for a degree or diploma.

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ACKNOWLEDGEMENTS

I wish to express my deep sense of appreciation and gratitude to my supervisor Professor R.S.L. Srivastava, Head of the Department of Mathematics, Indian Institute of Technology, Kanpur, for his valuable guidance and inspiring discussions. I take this opportunity to thank all the members of the Complex Function Theory Group of the Department of Mathematics for their helpful discussions and cooperation during the progress of the work. Thanks are also due to Sri G.L. Misra for the pains he took in typing out the manuscript.

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Dated: July 31 , 1970.

CONTENTS

SYNOPSIS

Univalent functions have been extensively studied during the last fifty to sixty years by different workers in this field. The failure to settle Bieberbach's conjecture in its generality has led to the investigation of various subclasses of univalent functions. Usually the following types of problems are studied for an univalent, regular and normalized function $f(z)$: (a) Distortion theorems, i.e., determination of lower and upper estimates for $|f(z)|$, $|f'(z)|$ and $|\frac{zf'(z)}{f(z)}|$, etc. (b) Coefficient estimates (c) Bounds for arg $\{\frac{f(z)}{z}\}$ and arg $\{f'(z)\}$ (d) Radii of starlikeness and convexity. Investigations similar to the problems (a), (b) and (d) are also carried out in the case of p-valent, mean p-valent and circumferentially mean p-valent functions.

The present thesis consists of seven chapters.

Chapter 1 is a brief introduction to some results in the theory of univalent functions and multivalent functions and describes the problems which have been investigated further in the remaining six chapters.

The converse problem of S. Bernardi (Trans. Amer. Math. Soc. 135 (1969), 429-446) has been studied in chapter 2 and sharp results have been obtained which, in particular, include the results of Padmnabhan (J. London Math. Soc., 1 (part 2), (1969), 226 - 231). In the same chapter a region of convexity for functions of the class $\overline{S}^*_\delta$, defined by D.J. Wright (Comp. Math. 20, (1969), 122 - 124), has been obtained and the results have been shown to be sharp. In particular, these include a result of D.J. Wright (which is also due to Ram Singh (Comp. Math. 21 (1969), 230 - 231)). How the radius of convexity of univalent starlike function of order β depends on the

coefficients of Taylor expansion around the origin, has also been investigated in the same chapter.

In chapter 3, the distortion theorems for the α-spiral univalent functions have been obtained. Some results are shown to be sharp. These results in particular include some results of Finkelstein (Proc. Amer. Math. Soc., 18 (1967), 412 - 418) and D.E. Tepper (Thesis, Temple University, U.S.A., (1968)).

Coefficient estimates for α-spiral univalent functions have been made in chapter 4. The results of this chapter include some results of Libera (Canadian J. Math., 19 (1967), 449 - 456), MacGregor (Michigan Math. J. 10 (1963), 277 - 281) and others.

Chapter 5 deals with the radius of convexity region for the class of generalized meromorphic close-to-convex functions, i.e., the class of $B(\beta,\lambda,\sigma)$ defined and introduced in this work. In particular, it generalizes a theorem of Libera. From the results obtained one gets the region of univalence for $f(z) \in B(\beta,\lambda,\sigma)$. Coefficient estimates for $f(z) \in B(\beta,\lambda,\sigma)$ have also been obtained.

Necessary and sufficient conditions on the measure function, occuring in the integral representation for the class $T^*(p)$ of p-valent typically real functions, have been obtained in chapter 6. The coefficient estimates for $f(z) \in T(p)$ have also been derived.

Finally, in chapter 7, coefficient estimate for bounded Bazilevič p-valent function has been obtained.

CHAPTER 1

INTRODUCTION

1.1. A function $f(z)$ is said to be univalent (or schlicht) in a domain D, if for any two points z_1 and z_2 of D we have $f(z_1) = f(z_2)$ only if $z_1 = z_2$. A function $f(z)$, which is regular and univalent in the unit disc $D\{|z| < 1\}$ may be normalized by the conditions $f(0) = 0$, $f'(0) = 1$. The normalization of a function is not an essential restriction, for , if $f(z)$ is univalent, so is the function $g(z) = \frac{f(z) - f(0)}{f'(0)}$. We shall denote by S the class of analytic functions $f(z)$ which are regular and univalent in the unit disc* D and which are normalized by the conditions $f(0) = 0$, $f'(0) = 1$. The Taylor expansion of such a function about the origin has the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.

The origin of the theory of univalent functions can be traced to a paper by P. Koebe in 1907 on the uniformization of algebraic curves [27]. In this paper Koebe proved in particular, that there is a constant k, (called Koebe's constant), such that the boundary of the map of $|z| < 1$ by any function $W = f(z)$ of the class S is always at a distance not less than k from $W = 0$. Koebe's work was followed by a number of other workers (Plemelj [48], Gronwall [19], Faber [10], Bieberbach [5], and others). In particular, Bieberbach in 1916 obtained that $k = \frac{1}{4}$, which could also be found from the results of Gronwall [19]. Bieberbach also found that $|a_2| \leq 2$ for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$. Since the equality

*Hereinafter we shall denote the unit disc $|z| < 1$ by D.

in the above result is attained for the function (Koebe function)

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z(1 + e^{i\phi}z)^{-2}$, ϕ real, and also because $|a_n| = n$,

$n = 2,3,\ldots$ for the above function, it was conjectured by Bieberbach that

$$(1.1.1) \qquad |a_n| \leq n\ , \quad n = 2,3,\ldots$$

for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$. Approximately at the same time Bieberbach proposed the coefficient problem. This is the problem of finding for each $n, n \geq 2$, the precise region V_n in Euclidean space of $2n-2$ real dimensions occupied by the points $(a_2, a_3, \ldots, a_n)$ corresponding to functions of class S. No satisfactory and complete solution for this problem is known so far. However, in particular for spaces V_2, V_3, V_4 the solution is known. Infact, Bieberbach [5], Löwner [35], Garabedian and Schiffer [12] and R.N. Pederson [47] have proved, respectively, that

$$(1.1.2) \qquad |a_n| \leq n \quad \text{for } n = 2,3,4,6.$$

No solution is known for a_5 and hence the coefficient problem for V_5 and V_6 is unsolved. Quite recently the problem of local maximality for the coefficients a_6 and a_8 has been considered by Jenkins and Ozawa ([23], [24] and Ozawa ([41], [42], [43], [44], [45]).

Various subclasses of the class of univalent functions have been investigated by different workers in this field. The fact that a family of analytic functions $f(z)$ which is normal, compact and satisfies the condition $|f'(c)| \geq \delta > 0$ at a point c in a domain D have a circle of univalence about c, follows from the theory of normal families [7].

Although several sufficient conditions for univalence of an analytic function are known ([37], [38], [57], [68], and others), exact determination of the radius of univalence of a given analytic function is often not easy.

An important subclass of the class S of univalent functions is the class of convex functions. A function f is said to be convex in D if image under f of the domain D is a convex set. We shall denote by C the class of functions convex in D. If further, all members of the class C are regular, univalent in the unit disc D and are normalized by the conditions $f(0) = 0$, $f'(0) = 1$, then C is a subclass of S. A necessary and sufficient condition for a function $f \varepsilon S$ to be convex has been given by M.S. Robertson [52]. It states that $f \varepsilon S$ is convex in $|z| \leq r$, if and only if,

(1.1.3) $$\mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\} \geq 0$$

for all z such that $|z| \leq r$. Robertson [52] has also defined the order β of a convex function $f \varepsilon S$. Thus, $f \varepsilon C$ is a convex function of order β in the unit disc D, if

(1.1.4) $$\mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\} \geq \beta \geq 0 \quad \text{for all } z \varepsilon D\{|z| < 1\},$$

and $0 \leq \beta < 1$ and if for every $\varepsilon > 0$, sufficiently small, there is a $z = z_o$, $|z_o| \leq 1$, for which

(1.1.5) $$\mathrm{Re}\{1+\frac{z_o f''(z_o)}{f'(z_o)}\} < \beta+\varepsilon$$

We shall denote this class of functions by C_β .

A wider subclass of S than the class C of convex functions is the class of starlike functions, usually denoted by S^*. A function $f \varepsilon S$ is said to be starlike in D with respect to $W = 0$, if any point $W = f(z)$ in the image set of D, when joined to the origin $W = 0$ by a line segment has the property that all points of such a line segment belong to the image set of D under f. Equivalently, $f \varepsilon S$ is starlike in D relative to origin if it is mapped into a domain which has the property that every straight line through the origin cuts the contour enclosing the domain in not more than two points. A necessary and sufficient condition for $f \varepsilon S$ to be starlike in D with respect to the origin is wellknown [52]. Thus a function $f \varepsilon S$ is starlike with respect to $W = 0$ in $|z| \leq r$, if and only if,

$$(1.1.6) \qquad \mathrm{Re}\{\frac{zf'(z)}{f(z)}\} \geq 0 \quad \text{for all } z \text{ such that } |z| \leq r$$

As in the case of convex functions, the order β has also been defined by Robertson [52] for the case of starlike functions. Thus $f \varepsilon S^*$ is said to be starlike function of order β, if

$$(1.1.7) \qquad \mathrm{Re}\{\frac{zf'(z)}{f(z)}\} \geq \beta \geq 0 \quad \text{for } |z| < 1,\ 0 \leq \beta \leq 1 ,$$

and, if for every $\varepsilon > 0$ sufficiently small there is a $z = z_o$, $|z_o| \leq 1$, for which

$$(1.1.8) \qquad \mathrm{Re}\{\frac{z_o f'(z_o)}{f(z_o)}\} < \beta+\varepsilon$$

We denote this class of functions by the symbol S_β^*. It is to be noted further from (1.1.4) and (1.1.7) that

$$(1.1.9) \qquad f(z) \in C_\beta, \text{ if and only if, } zf'(z) \in S_\beta^* .$$

Yet another useful subclass $\overline{S}_\delta^*$ of univalent starlike functions is due to D.J. Wright [67]. $f \in \overline{S}_\delta^*$, if and only if,

$$(1.1.10) \qquad \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \delta$$

for $z \in D$ and $0 \leq \delta \leq 1$.

A class wider than the class of starlike functinns is the class of spiral-like functions introduced by L. Špaček [62] in 1932. Špaček essentially showed that a function $f \in S$ is spiral-like in D if

$$(1.1.11) \qquad \mathrm{Re}\left\{\frac{\xi zf'(z)}{f(z)}\right\} \geq 0, \ |\xi| = 1, \ z \in D.$$

If we replace ξ by $e^{i\alpha}$, with $|\alpha| \leq \frac{\pi}{2}$, then f is called [32] univalent α-spiral function. We shall denote the class of univalent α-spiral functions by $S(\alpha)$. Clearly, starlike functions are also α-spiral functions with $\alpha = 0$. The order ρ of univalent α-spiral function has recently been introduced by Libera [32]. Thus, $f \in S(\alpha)$ is said to be of order ρ in D, if

$$(1.1.12) \qquad \mathrm{Re}\left\{\frac{e^{i\alpha}zf'(z)}{f(z)}\right\} \geq \rho \geq 0, \ 0 \leq \rho < 1, \quad \text{for all } z \in D.$$

According to Libera, we shall consider the functions belonging to the classes

S^*_β and C_β only with the conditions (1.1.7) and (1.1.4) and call β to be the order of the respective classes. These conditions are less restrictive than those referred on the pages 3 and 4.

Another class, wider than that of starlike functions is the class of close-to-convex univalent functions, which we shall denote by Γ. This class has been introduced by W. Kaplan [25]. If $f(z)$ be analytic in $|z| < 1$, then $f(z)$ is close-to-convex for $|z| < 1$ if there exists a function $\phi(z)$, convex and univalent for $|z| < 1$, such that $\frac{f'(z)}{\phi'(z)}$ has positive real part for $|z| < 1$. This class of functions in particular includes the class S^*. Kaplan [25] further showed that if $f(z)$ is close-to-convex then it is univalent. He further characterized close-to-convex functions, without reference to a convex function ϕ. Thus, $f(z)$ is close-to-convex, if and only if,

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)}\right\} d\theta > -\pi \tag{1.1.13}$$

where $\theta_1 < \theta_2$, $z = re^{i\theta}$ and $r < 1$.

The order β and type λ for $f(z) \varepsilon \Gamma$ has been introduced by Libera [29]. If $f(z)$ be analytic with $f(0) = 0$, $f'(0) = 1$ in the unit disc D and β and λ lie in the closed interval $[0,1]$, then $f(z)$ is said to be close-to-convex of order λ and type β if and only if for some $F(z) \varepsilon S^*_\beta$ we have

$$\mathrm{Re}\left\{\frac{zf'(z)}{F(z)}\right\} \geq \lambda, \text{ for all } z \varepsilon D \tag{1.1.14}$$

We denote this class of functions by $\Gamma(\lambda,\beta)$ and identify $\Gamma(0,0) = \Gamma$.

In fact, from the above definitions it follows that

$$(1.1.15)\ \begin{cases} C \subset S^* \subseteq \Gamma \subset S,\ C \subseteq S^* \subseteq S(\alpha) \subset S,\ S^*_\beta \subseteq S^*,\ C_\beta \subseteq C \\ \text{and}\quad S_\rho(\alpha) \subseteq S(\alpha)\ ,\ \Gamma(\lambda,\beta) \subseteq \Gamma(0,0) \end{cases}$$

Some analogous extensions ([22], [31], [38]) of the classes S^*_β , $C_\beta, \Gamma(\lambda,\beta)$, $S_\rho(\alpha)$ are also carried over to the meromorphic univalent functions which are regular in the unit disc except at the point $z = 0$. Here we shall adopt the following definition given by Libera [31] which also includes the definition of Libera and Robertson [28] for a subclass of meromorphic close-to-convex functions of order λ and type σ. Denote by $B(\lambda,\sigma)$, $0 \leq \lambda,\sigma \leq 1$, the family of functions

$$(1.1.16)\qquad f(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + \ldots$$

which are regular in $E(0 < |z| < 1)$ and together with some function $F(z)$ having the representation

$$(1.1.17)\qquad F(z) = \frac{e^{i\alpha}}{z} + b_0 + b_1 z + b_2 z^2 + \ldots\ ,\ \alpha \text{ real}$$

regular in the annulus $E(0 < |z| < 1)$ and belonging to the class Σ^*_σ of meromorphically starlike functions of order σ, $0 \leq \sigma \leq 1$, i.e., $F \in \Sigma^*_\sigma$, if and only if,

$$(1.1.18)\qquad \operatorname{Re}\left\{\frac{-zF'(z)}{F(z)}\right\} \geq \sigma\ ,\ z \in E.$$

and

$$\text{(1.1.19)} \qquad \operatorname{Re}\left\{\frac{zf'(z)}{F(z)}\right\} \geq \lambda \, , \; z \in E.$$

If $f \in B(\lambda,\sigma)$ then we say "f is meromorphically close-to-convex of order λ and type σ"; and $f \in B(\lambda,\sigma)$ with respect to F is read "f is close-to-convex of order λ and type σ with respect to F. It is to be noticed [31] that if $f \in B(\lambda,\sigma)$, then it need not be univalent. In general, if $F(z) = z(1 + z^2)^{-(1-\sigma)}$ and $P(z) = \frac{1 + (1 - 2\lambda)z^2}{1 - z^2}$, then $F \in \Sigma^*_\sigma$ and $\operatorname{Re}\{P(z)\} \geq \lambda$, $z \in E$. Further, if $-zf'(z) = F(z).\ P(z)$, then

$$\text{(1.1.20)} \qquad f(z) = \frac{1}{z} - (3 - 2\lambda - \sigma)\, z + \ldots \in B(\lambda,\sigma).$$

Consequently, f is not univalent [31] for those values of λ and σ for which $(3 - 2\lambda - \sigma) > 1$.

The class $B(\lambda,\sigma)$ can be further extended. We denote the class $B(\beta,\lambda,\sigma)$, $0 \leq \lambda, \sigma < 1$, $\beta \geq 1$ as a family of functions $F(z)$ and $f(z)$ having representations (1.1.16) and (1.1.17) respectively, such that

$$\text{(1.1.21)} \qquad \left|\left[\left\{\frac{-zf'(z)\sec\alpha}{F(z)} + i\tan\alpha + \lambda\sec\alpha\right\}\{1 + \lambda\sec\alpha\}^{-1}\right] - \beta\right| < \beta$$

for $|z| < 1$. If $f \in B(\beta,\lambda,\sigma)$ then we shall call f to be memorphically generalized close-to-convex function.

In the context of coefficient problem yet another subclass of univalent functions of S was introduced by Rogosinski [58], who called the class as

a class of typically real functions, since all the coefficients of such a function in its Taylor expansion about the origin are taken to be real. We denote this class by $T(1)$. Thus, $f \varepsilon T(1)$ implies that f is regular and univalent in $|z| < 1$, has expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ with the property that f assumes real values if and only if z is real. For such a function f, $\operatorname{Im}\{f(z)\}$ and $\operatorname{Im}\{z\}$ always have the same sign in $|z| < 1$.

It must be pointed out here that Bieberbach's conjecture for the coefficients is true ([6], [20], [62], [51], [([9], [58])]) for functions belonging to the classes C, S^*, $S(\alpha)$, Γ and $T(1)$.

The concept of univalence can be extended to p-valence, mean-p-valence or circumferentially mean p-valence for analytic functions regular in the unit disc $D\{|z| < 1\}$. A regular function $f(z)$ in the unit disc $D\{|z| < 1\}$ is said to be p-valent, mean p-valent and circumferentially mean-p-valent if the following are satisfied respectively:

(1.1.22) The equation $f(z) = w$ has never more than p-solutions and there also exists a $z = z_o \varepsilon D$ such that $f(z) = w$ has exactly p-solutions.

(1.1.23) $W(1) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 n(\rho e^{i\phi}) \rho d\rho d\phi \leq p$ for $0 < |z| < 1$, where $n(\rho e^{i\phi})$ denotes the number of roots of the equation $f(z) = \rho e^{i\phi}$ in the disc $|z| < \rho$, and

$$(1.1.24) \qquad p(1) = \frac{1}{2\pi} \int_0^{2\pi} n(e^{i\phi}) d\phi \leq p$$

where $n(e^{i\phi})$ denotes the number of roots of the equation $f(z) = e^{i\phi}$ in the unit disc $D(|z| < 1)$.

In particular, the classes S^* and $T(1)$ have also extensions to the above class of p-valent or mean-p-valent functions. M.S. Robertson [54] has extended the class $T(1)$ to the class $T(p)$. A function $f(z) = \sum_{n=1}^{\infty} c_n z^n \varepsilon T(p)$ if and only if (a) $f(z)$ is regular in the unit disc $|z| < 1$ and all the coefficients are real., (b) there is a number $\delta = \delta(f)$, $0 < \delta < 1$, such that for each r in the interval $1 - \delta < r < 1$, $\mathrm{Im}\{f(z)\}$ changes sign 2p-times on the circle $|z| = r$.

In particular, the class $T(p)$ includes the class of p-valent real starlike functions which form a subclass of the class of p-valent regular starlike functions. According to Robertson [56], a regular function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ in the unit disc D is said to be p-valent starlike if and only if

$$(1.1.25) \qquad \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad \text{and} \quad \int_0^{2\pi} \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = 2\pi p$$

for $z = re^{i\theta}$, $\rho < r < 1$. Geometrically this means, that for a range $\rho < r < 1$, the image curve c_r of $|z| = r$, through the mapping $W = f(z)$, has the property that the vector joining the origin to the point $f(z)$ turns continuously through an angle $2\pi p$ in the anti-clockwise direction as z traverses the circle $|z| = r$ once in the same direction.

We define here another class $B(p)$ of p-valent regular functions which we call Bazilevič-p-valent regular functions. Thus $f \varepsilon B(p)$, if and only if,

$$(1.1.26) \qquad \mathrm{Re}\left\{\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right\} \geq \rho > 0$$

for some p-valent starlike regular function $g(z)$ in $|z| < 1$ and $\beta \geq 0$.

Similar to Bieberbach's conjecture for univalent functions, A.W. Goodman [17] conjectured that if $F(z) = \sum_{n=1}^{\infty} a_n z^n$ be regular and p-valent in D, then for $n > p$

$$(1.1.27) \qquad |a_n| \leq \sum_{k=1}^{p} \frac{2k\ (p+n)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)} |a_k|.$$

For $p = 1$, this conjecture reduces to the conjecture of Bieberbach for univalent functions. Goodman and Robertson [18] showed that (1.1.27) holds for all $n > p$ whenever $f(z)$ is p-valent starlike in D and the coefficients a_k are real. If the coefficients are complex then Robertson [56] proved (1.1.27) for $p = 2$. Very recently Livingston [34] has shown that (1.1.27) is also true for all n, if $F(z) = a_{p-1} z^{p-1} \ldots$, $|z| < 1$, and $F(z)$ is p-valent close-to-convex in D.

1.2. Usually the following types of problem are studied for univalent functions:

(a) Distortion theorems, i.e., determination of lower and upper estimates for $|f(z)|$, $|f'(z)|$, $\left|\frac{zf'(z)}{f(z)}\right|$.

(b) Coefficient estimates for $f(z) \varepsilon\ S$.

(c) Bounds for $\arg \{\frac{f(z)}{z}\}$ and $\arg \{f'(z)\}$.

(d) Radii of starlikeness and convexity for $f(z)$.

Investigations similar to (a), (b) and (d) have also been carried out in the case of p-valent, mean p-valent and circumferentially mean p-valent functions.

In the present work, we have mainly restricted our investigations to the problems of the type (a), (b) and (d), for certain subclasses of univalent functions and to the problem (b) and the area growth estimates for p-valent functions. We have also investigated problems (b) and (d) for generalized meromorphically close-to-convex functions.

In chapter 2, we have obtained the converses of the theorems of Bernardi [4]. The results of this chapter include the results of Padmnabhan [46] and are sharp. We have also obtained the region of radius of convexity for the class of functions $\overline{S}^*_\delta$ which includes a result of Ram Singh [60], which is also due to D.J. Wright [67]. In the last section of this chapter we have obtained a radius of convexity theorem for the class S^*_β which depends on the second coefficient $\frac{1}{2}f''(0)$. In particular, we have obtained the regions of convexity for S^*_0 when $f''(0) = 0$ and $S^*_{\frac{1}{2}}$ when $f''(0) = 0$.

In chapter 3, we have obtained distortion theorems for functions belonging to the class $S_\rho(\alpha)$, which include some of the results of Finkelstein [11] and Tepper [65]. Some of the results are also sharp.

In chapter 4, we have obtained the coefficient estimates for functions of the class $S_o(\alpha)$ and as an important consequence, we have obtained that $\frac{1}{n!}|f''(0)| \leq n \cos \alpha$, $n \geq 2$ which is a better bound than that obtained by Špaček. It is to be pointed out that these results have been obtained by the use of Herglotz's theorem, while Libera [32] has obtained the coefficient estimate by using the method of Clunie [8]. Further, in the same chapter, we have obtained a general coefficient estimate for $S_\rho(\alpha)$ which includes the above result of Libera and also includes some results of MacGregor [36] Golusin [15] and others by using the method of Clunie.

In chapter 5, we have obtained the radius of convexity region for the class $B(\beta,\lambda,\sigma)$ which also includes a theorem of Libera [31] in particular. From this result we also get the region of univalence for $f(z) \varepsilon B(\beta,\lambda,\sigma)$. In the same chapter we have found the coefficient estimates for $f(z) \varepsilon B(\beta,\lambda,\sigma)$.

In chapter 6, we have obtained some necessary and sufficient conditions on the measure function occuring in the integral representation [13] for functions of class $T^*(p)$. In particular, we have deduced the coefficient estimates for $f(z) \varepsilon T(p)$.

In chapter 7, the coefficient estimate for bounded Bazilevič p-valent functions has been obtained.

CHAPTER 2

The Radius Of Convexity And Starlikeness

2.1. In this chapter we study the following classes of univalent functions:

Definition 1. Let S denote the class of functions $W = f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are regular and univalent in the unit disc $D\{|z| < 1\}$.

Definition 2. Let S^* denote the class of functions $f(z)$ in S, which map D onto a region which is starlike with respect to $W = 0$.

The equivalent characterization of definition 2, as given by Robertson [52], states that $f(z)$ is starlike univalent function with respect to $W = 0$, if and only if,

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq 0 \quad \text{for} \quad |z| < 1.$$

Definition 3. Let S^*_β denote the class of functions $f(z)$ in S^* which have the additional property that

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \beta \ ; \ z \in D$$

where $0 \leq \beta \leq 1$.

Here β is referred as the order of the starlike univalent function $f(z)$, and we identify $S^*_0 \equiv S^*$.

Definition 4. Let $\overline{S^*_\delta}$ denote the subclass of functions $f(z)$ of S^*_0 for which

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \delta$$

for $z \in D$.

Definition 5. Let C denote the class of functions $f(z)$ in S which map D onto a region which is convex.

Analytically, this definition can be equivalently reformulated by the following characterization given by Robertson [52].

Definition 5'. A function $f(z)$ is convex univalent if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq 0$$

for all $z \in D$.

Definition 6. Let C_β denote the class of convex univalent functions of order β if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \beta \ , \ z \in D$$

where $0 \leq \beta \leq 1$.

Definition 7. If $f(z) \in S$ and $g(z) \in S^*_\beta$ satisfy the conditions

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} \geq \lambda \quad \text{and} \quad \operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geq \beta$$

for all $z \in D$ and $0 \leq \lambda, \beta \leq 1$, then $f(z)$ is said to be close-to-convex function with respect to $g(z)$ of order λ and type β.

We denote this class by $\Gamma(\lambda,\beta)$.

Definition 8. If $\lambda = 0$ and $\beta = 0$ then $f(z)$ is simply said to be close-to-convex function with respect to $g(z)$.

This concept of close-to-convex functions with respect to another function was introduced by W. Kaplan [25] and its extension appears in the works of Robertson, Libera and Robertson and others.

Libera [30] in 1965 established the following theorems :

Theorem A[Libera] : If $f \in S^*$ (or $f \in C$) then the function $F(z) = \frac{2}{z}\int_0^z f(t)\,dt \in S^*$(or $F \in C$).

Theorem B[Libera] : If f is close-to-convex with respect to g,

$$F(z) = \frac{2}{z}\int_0^z f(t)\,dt \text{ and } G(z) = \frac{2}{z}\int_0^z g(t)\,dt$$

then F is close-to-convex with respect to G.

S. Bernardi [3] extends theorems A and B, and he proves in general the following:

Theorem C[Bernardi] : If $f(z) \in S^*$ (or C) then

$$g(z) = \sum_{n=1}^{\infty} \left(\frac{c+1}{c+n}\right) a_n z^n = (c+1)\, z^{-c} \int_0^z t^{c-1} f(t)\,dt \quad \text{with} \quad a_1 = 1$$

and $c = 1, 2, 3, \ldots$, then $g(z) \in S^*$.

Theorem D[Bernardi] : Let $f(z)$ be close-to-convex with respect to $g(z)$, $c = 1, 2, 3, \ldots$

$$f(z) = \left(\frac{1}{1+c}\right) z^{1-c}[z^c F(z)]'; \; g(z) = \left(\frac{1}{1+c}\right) z^{1-c}[z^c G(z)]'$$

Then $F(z)$ is close-to-convex with respect to $G(z)$.

The converse problem of Libera [30] is treated by Livingston [33] who proves the following :

Theorem E[Livingston] : If $F(z)$ is in S^* then $f(z) = \frac{1}{2}[z\, F(z)]'$ is starlike for $|z| < \frac{1}{2}$. This result is sharp.

Theorem F[Livingston]: If F is in C, then $f(z) = \frac{1}{2}[z\, F(z)]'$ is univalent in D and is convex for $|z| < \frac{1}{2}$. This result is also sharp.

Bernardi [4] again considered the converse problem of theorems C and D and he proves the following :

Theorem G[Bernardi] : If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$,

$$g(z) = \sum_{n=1}^{\infty} \left(\frac{c+1}{c+n}\right) a_n z^n = (c+1)\, z^{-c} \int_0^z t^{c-1} f(t)\, dt,$$

with $a_1 = 1$ and $c = 1,2,3,\ldots$, and $g(z) \varepsilon S^*$ (or C) then $f(z)$ is starlike (or convex) in the region $|z| < \frac{-2+(3+c^2)^{1/2}}{c-1}$ for $c = 2, 3, 4, \ldots$ and $|z| < \frac{1}{2}$ for $c = 1$. This result is sharp.

Theorem H [Bernardi] : Let $F(z)$ be close-to-convex with respect to $G(z) \varepsilon S^*$ and

$$f(z) = \frac{1}{1+c}\, z^{1-c}[z^c\, F(z)]',$$

$$g(z) = \frac{1}{1+c}\, z^{1-c}[z^c\, G(z)]'$$

then $f(z)$ is close-to-convex with respect to $g(z)$ in the region

$|z| < \frac{-2+\sqrt{3+c^2}}{c-1}$ for $c = 2, 3, 4, \ldots$, and $|z| < \frac{1}{2}$, if $c = 1$.

K.S. Padmanabhan [46] considered the converse problem of Libera [30] for the class S^*_β, C_β and $\Gamma(\rho,\beta)$. In this chapter we are mainly concerned with the radius of starlikeness and radius of convexity for functions S^*_β, $\overline{S}^*_\delta$

and C_β respectively. In particular we derive the converses of theorems G and H of Bernardi for the classes S^*_β, C_β and $\Gamma(\lambda,\beta)$. We notice that these results are sharp. With these extensions we deduce the theorems of Padmnabhan also. Incidently, the proof of our theorem 1,which includes a theorem of Padmnabhan, is much simpler and can also be adopted for the restricted case considered by him.

In section 3, we consider the problem of radius of convexity for the class $\overline{S}^*_\delta$ which includes a result of Ram Singh [60] and which is also due to D.J. Wright [67].

We shall need the following lemmas :

Lemma 1 [39] : Let the analytic function $f(z)$ be regular in the unit disc $|z| < 1$ and let $f(0) = 0$. If, in $|z| < 1$, $|f(z)| \le 1$, then

$$|f(z)| \le |z| \quad , \quad |z| < 1$$

where equality can hold only if $f(z) \equiv Kz$ and $|K| = 1$.

The following lemma is known as generalized Schwarz's lemma.

Lemma 2([39], p. 167): If $f(z)$ be regular in the disc $D(|z| < 1)$ and $|f(z)| \le 1$ there, then

$$\frac{|f(0)| - |z|}{1 - |f(0)||z|} \le |f(z)| \le \frac{|z| + |f(0)|}{1 + |f(0)||z|}$$

The following lemma is due to S. Bernardi :

Lemma 3 ([3], p.314) : Let $f(z)$ and $g(z)$ be regular in $|z| < 1$, $g(z)$ map $|z| < 1$ onto a many sheeted starlike region, α,β real, $\text{Re}\{e^{i\beta}\frac{f'(z)}{g'(z)}\} > \alpha$ for $|z| < 1$. $g(0) = f(0) = 0$. Then $\text{Re}\{e^{i\beta}\frac{f(z)}{g(z)}\} > \alpha$ for $|z| < 1$.

Further, if $f(z) \in S^*$, $g(z) = \int_0^z H(t)\,dt = \int_0^z t^{p-1} f(t)\,dt$, then $g(z)$ is $(p+1)$-valent starlike for $p = 1, 2, 3, \ldots$.

The following result is due to D.J. Wright [67], which we state as a lemma.

Lemma 4 [67] : $f(z) \in \overline{S}^*_\delta$, if and only if, $f(z) = z \exp\left[(1-\delta)\int_0^z \phi(t)\,dt\right]$ where $\phi(z)$ is regular and bounded by 1 in the unit disc D.

2.2 We shall prove the following :

Theorem 1 : If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*_\beta$ and $g(z) = \sum_{n=1}^{\infty} \left(\frac{c+1}{c+n}\right) a_n z^n$ $= (c+1) z^{-c} \int_0^z t^{c-1} f(t)\,dt$, with $a_1 = 1$ and $c = 1,2,3,\ldots$, then $g(z) \in S^*_\beta$ and conversely, if $g(z) \in S^*_\beta$ then $f(z)$ is starlike of order β in the region

$$|z| < r_0 = \begin{cases} \dfrac{-(2-\beta) + (3+\beta^2+c^2+2c\beta-2\beta)^{1/2}}{(c+2\beta-1)}, & \text{if } c = 2, 3, \ldots \\[2ex] \dfrac{1}{2}, & \text{if } c = 1 \text{ and } \beta = 0 \\[2ex] \dfrac{-(2-\beta) + (4+\beta^2)^{1/2}}{2\beta}, & \text{if } c = 1 \text{ and } 0 < \beta < 1 \end{cases}$$

Remark : The essential ideas in the first part of the proof are the same as given by Bernardi.

Proof : Let

$$J(z) = \int_0^z t^{c-1} f(t)\,dt$$

A direct computation yields that

$$J'(z) = z^{c-1} f(z)$$

$$g'(z) = (c+1)\{z^{-1} f(z) - cz^{-c-1} \int_0^z t^{c-1} f(t)\, dt\}$$

and

$$[z^{c+1} g'(z)]' = (c+1)[cz^{c-1} f(z) + z^c f'(z) - cz^{c-1} f(z)] = (c+1)z^c f'(z)$$

Hence,

$$\text{(2.2.1)} \qquad \mathrm{Re}\left\{\frac{[z^{c+1} g'(z)]'}{J'(z)}\right\} = (c+1)\, \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq (c+1)\,\beta$$

Therefore, by lemma 3 and (2.2.1), we have

$$\mathrm{Re}\left\{\frac{z^{c+1} g'(z).(c+1)}{g(z) z^c}\right\} \geq \beta(c+1)$$

This implies that

$$\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geq \beta$$

and the proof of first part of the theorem is complete.

Conversely, by the hypothesis of the theorem

$$g(z) = \frac{(c+1)}{z^c} \int_0^z t^{c-1} f(t)\, dt$$

Hence,

$$\text{(2.2.2)} \quad \frac{zg'(z)}{g(z)} = \left[\frac{(c+1)z^{c-1} f(z)}{z^c} + \frac{(-c)(c+1)}{z^{c+1}} \int_0^z t^{c-1} f(t)dt\right] \Big/ \frac{(c+1)}{z^c} \int_0^z t^{c-1} f(t)dt$$

$$= \frac{1}{z}[(c+1)\{f(z) - \frac{c}{z^c}\int_0^z t^{c-1} f(t)\, dt\}]/\frac{(c+1)}{z^c}\int_0^z t^{c-1} f(t)\, dt$$

$$= \frac{zJ' - cJ}{J}$$

Further, since $g(z)$ is starlike function of order β, so there exists a function $\omega(z)$ which is regular in the unit disc D and satisfies the conditions of Schwarz's lemma, such that

$$\frac{zg'(z)}{g(z)} = \frac{1-(1-2\beta)\omega(z)}{1+\omega(z)} \tag{2.2.3}$$

From (2.2.2) and (2.2.3) it follows that

$$\frac{zJ' - cJ}{J} = \frac{1 + (2\beta-1)\omega(z)}{1+\omega(z)}$$

or
$$z^c f(z) = \frac{[1 + (2\beta-1)\omega(z) + c + c\omega(z)]\, J}{1+\omega(z)}$$

or
$$f(z) = \frac{[(1+c) + \{c + (2\beta-1)\}\, \omega(z)]\, J}{[1+\omega(z)]\, z^c}$$

Differentiating logarithmically we obtain

$$\frac{zf'(z)}{f(z)} = \frac{z[c + 2\beta-1]\, \omega'(z)}{1+c+[c+2\beta-1]\omega(z)} + \frac{zJ' - cJ}{J} - \frac{z\omega'(z)}{1+\omega(z)}$$

Using (2.2.2) and (2.2.3) we obtain

$$(2.2.4)\qquad \frac{zf'(z)}{f(z)}-\beta=\frac{z\omega'(z)[(c+2\beta-1)(1+\omega(z))-\{(1+c)+c\omega(z)+(2\beta-1)\omega(z)\}]}{[1+\omega(z)][(1+c)+(c+2\beta-1)\omega(z)]}$$

$$+\frac{1+(2\beta-1)\omega(z)}{1+\omega(z)}-\beta-\beta\omega(z)$$

$$=\frac{z\omega'(z)[(c+2\beta-1-1-c)+(c+2\beta-1-c-2\beta+1)\omega(z)]}{[1+\omega(z)][(1+c)+(c+2\beta-1)\omega(z)]}+\frac{(1-\beta)(1-\omega(z))}{1+\omega(z)}$$

$$=(1-\beta)\left[\frac{1-\omega(z)}{1+\omega(z)}-\frac{2z\omega'(z)}{[1+\omega(z)][(1+c)+(c+2\beta-1)\omega(z)]}\right]$$

But

$$(2.2.5)\qquad \mathrm{Re}\left\{\frac{1-\omega(z)}{1+\omega(z)}\right\}=\frac{1-|\omega(z)|^2}{|1+\omega(z)|^2}$$

and

$$(2.2.6)\qquad \mathrm{Re}\left\{\frac{2z\omega'(z)}{[1+\omega(z)][(1+c)+(c+2\beta-1)\omega(z)]}\right\}$$

$$\leq\frac{2|z||\omega'(z)|}{|1+\omega(z)||(1+c)+(c+2\beta-1)\omega(z)|}$$

$$\leq\frac{2|z|(1-|\omega(z)|^2)}{(1-|z|^2)|1+\omega(z)||(1+c)+(c+2\beta-1)\omega(z)|}$$

The last inequality has been obtained by using the following wellknown inequality ([39], p. 168).

$$(2.2.7)\qquad |\omega'(z)|\leq\frac{(1-|\omega(z)|^2)}{(1-|z|^2)}$$

Thus from (2.2.4) we note that $f(z)$ is starlike of order β, if

$$\frac{2|z|(1-|\omega(z)|^2)}{|1+\omega(z)|(1-|z|^2)|(1+c)+(c+2\beta-1)\omega(z)|} \le \frac{1-|\omega(z)|^2}{|1+\omega(z)|^2}$$

or
$$\frac{2|z|}{(1-|z|^2)|(1+c)+(c+2\beta-1)\omega(z)|} \le \frac{1}{|1+\omega(z)|}$$

or

$$\frac{2|z|}{1-|z|^2} \le \frac{(1+c)\left|1+\frac{c+2\beta-1}{1+c}\omega(z)\right|}{|1+\omega(z)|} \tag{2.2.8}$$

Since $|\omega(z)| \le |z|$ and $\frac{c+2\beta-1}{1+c} \le 1$, we have

$$\frac{\left|1+\frac{c+2\beta-1}{1+c}\omega(z)\right|}{|1+\omega(z)|} \ge \frac{1+\frac{(c+2\beta-1)}{(1+c)}|z|}{1+|z|} \tag{2.2.9}$$

Hence, by (2.2.8) and (2.2.9), we obtain that $f(z) \in S^*_\beta$ if

$$\frac{2|z|}{1-|z|^2} \le \frac{(1+c)+(c+2\beta-1)|z|}{1+|z|}$$

or
$$2|z| \le \{(1+c)+(c+2\beta-1)|z|\}(1-|z|)$$

i.e.
$$(1+c)-(2+1+c-c-2\beta+1)|z|-(c+2\beta-1)|z|^2 > 0$$

or
$$(1+c)-2(2-\beta)|z|-(c+2\beta-1)|z|^2 > 0$$

Let $P(r) = (1+c) - 2(2-\beta)r - (c+2\beta-1)r^2$. Since $P(0) = 1+c$ and

$P'(r) = -2(2-\beta) -2(c+2\beta-1)r < 0$, so $P(r)$ is a decreasing function of r. Thus the positive root r_o for which $P(r) > 0$ must be less than the root of the polynomial

$$(1+c) - 2(2-\beta)r - (c+2\beta-1)r^2 = 0$$

i.e.,
$$r_o = \frac{-(2-\beta) + \sqrt{4+\beta^2-4\beta+c+2\beta-1+c^2+2c\beta-c}}{(c+2\beta-1)}$$

$$= \frac{-(2-\beta) + \sqrt{3+\beta^2+c^2+2\beta(c-1)}}{(c+2\beta-1)}$$

This gives the required value of r_o and the proof of theorem 1 is complete.

The following example shows that the result of theorem 1 is sharp for each c.

Example. Consider the function

$$f(z) = z(1-z)^{2(1-\beta)}, \quad 0 \leq \beta < 1$$

Clearly $f(z) \in S^*_\beta$. By direct computation we obtain

$$\frac{zf'(z)}{f(z)} - \beta = \frac{(1-\beta)(1-z) + (3-2\beta)z}{1-z} - \frac{z(c+2\beta-1)}{(1+c)-(c+2\beta-1)z}$$

$$= \frac{(1-\beta)\ [(1+c) + 2(2-\beta)\ z-(c+2\beta-1)z^2]}{(1-z)[(1+c) - (c+2\beta-1)z]}$$

Thus $\frac{zf'(z)}{f(z)} - \beta = 0$ for $z = -r_o$. Hence $f(z)$ is not starlike in any circle $|z| < r$, if $r > r_o$.

The theorem [G] of Bernardi now follows by taking $c = 1,2,3,\ldots$ and $\beta = 0$.

If we take $c = 1$ and $\beta \geq 0$, then the following theorem of Padmnabhan [46] follows as a corollary to theorem 1.

Theorem [Padmnabhan]. Let $g(z) \in S^*_\beta$. Then $f(z) = \frac{1}{2}(zg(z))'$ is starlike of order β for

$$|z| < \{(\beta-2) + \sqrt{\beta^2+4}\}/2\beta.$$

Theorem 2. If $g(z) \in C_\beta$, $f(z) = \frac{1}{1+c} z^{1-c} [z^c g'(z)]'$, $c = 1,2,3,\ldots$, then $f(z)$ is convex for

$$|z| < \begin{cases} \dfrac{-(2-\beta) + \sqrt{3+\beta^2+c^2+2c\beta-2\beta}}{(c+2\beta-1)}, & \text{if } c = 2,3,4,\ldots . \\[2ex] \dfrac{1}{2}, & \text{if } c = 1 \text{ and } \beta = 0 \\[2ex] \dfrac{-(2-\beta) + \sqrt{4+\beta^2}}{2\beta}, & \text{if } c = 1 \text{ and } 0 < \beta < 1 \end{cases}$$

and the result is sharp.

Proof : Proof of theorem follows immediately by using the fact that if $g(z) \in C_\beta$ then $zg'(z) \in S^*_\beta$ and conversely (see 1.1.9). Therefore by theorem 1, the result follows.

The following example shows that the result of theorem 2 is sharp.

Example 2 : Let $\mathcal{F}(z) = 1 - \frac{(1-z)^{2\beta-1}}{2\beta-1}$, $\beta \neq \frac{1}{2}$ and if $\beta = \frac{1}{2}$ then $\mathcal{F}(z) = -\log(1-z)$.

By direct computation we find that

$$f(z) = \frac{z^{1-c}}{1+c}\left[z^c - \frac{z^c(1-z)^{2\beta-1}}{2\beta-1}\right]',$$

$$= \frac{z^{1-c}}{1+c}\left[cz^{c-1} - \frac{cz^{c-1}(1-z)^{2\beta-1}}{2\beta-1} + \frac{z^c(1-z)^{2\beta-2}(2\beta-1)}{(2\beta-1)}\right]$$

$$= \frac{1}{1+c}\left[c - \frac{c(1-z)^{2\beta-1}}{(2\beta-1)} + z(1-z)^{2\beta-2}\right]$$

$$= \frac{1}{(1+c)(2\beta-1)}\left[(2\beta-1)c-c(1-z)^{2\beta-1} + (2\beta-1)z(1-z)^{2\beta-2}\right]$$

$$f'(z) = \frac{(1-2\beta)(2\beta-2)z(1-z)^{2\beta-3}+c(2\beta-1)(1-z)^{2\beta-2} + (2\beta-1)(1-z)^{2\beta-2}}{(1+c)(2\beta-1)}$$

$$= \frac{(1+c)(2\beta-1)(1-z)^{2\beta-2} - (2\beta-1)(2\beta-2)\; z(1-z)^{2\beta-3}}{(1+c)\;(2\beta-1)}$$

$$= \frac{(1-z)^{2\beta-3}(2\beta-1)}{(1+c)\;(2\beta-1)}\left[(1+c)\;(1-z) - (2\beta-2)\; z\right]$$

$$= \frac{(1-z)^{2\beta-3}}{1+c}\left[(1+c) + (-1-c+2-2\beta)z\right]$$

$$= \frac{(1-z)^{2\beta-3}}{(1+c)}\left[(1+c) + (1-c-2\beta)\; z\right]$$

Differentiating logarithmically, we have

$$\frac{f''(z)}{f'(z)} = -\left(\frac{2\beta-3}{1-z}\right) + \frac{(1-c-2\beta)}{(1+c)+(1-c-2\beta)z}$$

This implies that

$$\frac{zf''(z)}{f'(z)} + 1-\beta = -\frac{(2\beta-3)z}{(1-z)} + \frac{(1-c-2\beta)z}{(1+c)+(1-c-2\beta)z} + 1 - \beta$$

$$= -\frac{(2\beta-3)z}{1-z} + \frac{(1+c)(1-\beta)+z[(1-\beta)(1-c-2\beta)+(1-c-2\beta)]}{(1+c) + (1-c-2\beta)z}$$

$$= \frac{(1+c)(1-\beta)+z[(3-2\beta)(1+c)-(1+c)(1-\beta)+(2-\beta)(1-c-2\beta)]+z^2[(1-c-2\beta)(-2\beta+3-2+\beta)]}{(1-z)[(1+c)+(1-c-2\beta)z]}$$

$$= \frac{(1+c)(1-\beta)+z[(1+c)(3-2\beta-1+\beta)+(2-\beta)(1-c-2\beta)]+z^2[(1-c-2\beta)(1-\beta)]}{(1-z)[(1+c)+(1-c-2\beta)z]}$$

$$= \frac{(1-\beta)[(1+c)+2(2-\beta)z-(c+2\beta-1)z^2]}{(1-z)[(1+c) - (c+2\beta-1)z]}$$

Thus the expression $(1-\beta) + \frac{zf''(z)}{f'(z)}$ vanishes for $z = -r_o$ hence $f(z)$ is not convex of order β in any circle $|z| < r, r > r_o$. Similarly for $\beta = \frac{1}{2}$ and $F(z) = -\log(1-z)$, sharpness of the theorem can be established.

It we take $c = 1$ and $0 \leq \beta \leq \frac{1}{2}$ in theorem 2, then the following theorem of Padmnabhan [46] is obtained as a corollary to theorem 2.

Theorem [Padmnabhan]. Let $g(z) \in C_\beta$. Then $f(z) = \frac{1}{2}(zg(z))' \in C_\beta$ for $|z| < \{(\beta-2) + \sqrt{\beta^2+4}\}/2\beta$.

For this class of convex functions, if $\beta = 0$, then theorem G of Bernardi follows as a corollary to theorem 2.

Theorem 3 : Let $f(z)=\frac{z^{1-c}}{1+c}[z^cF(z)]'$ and $g(z) = \frac{z^{1-c}}{1+c}[z^c G(z)]'$, $c = 1, 2, 3, \ldots$, $G(z) \in S^*_\beta$, and $F(z) \in \Gamma(\lambda,\beta)$ with respect to $G(z)$. Then $f(z)$ is close-to-convex with respect to $g(z)$ of order λ and type β with respect to the associated function $g(z)$ for

$$|z| < r_o = \begin{cases} \dfrac{-(2-\beta) + \sqrt{3+\beta^2+c^2+2c\beta-2\beta}}{(c+2\beta-1)}, & \text{if } c = 1,2,3,\ldots,\ \beta \geq 0 \\ \text{and } \beta \text{ is chosen such that } c+2\beta-1 > 0 & \\ \dfrac{1}{2}, \text{ if } c = 1 \text{ and } \beta = 0 & \end{cases}$$

This result is sharp.

Proof : Since $F \in \Gamma(\lambda,\beta)$, therefore there exists a function $G(z) \in S^*_\beta$ such that for $|z| < 1$

$$\text{Re}\ \{\frac{zF'(f)}{G(z)}\} > \lambda \text{ and } \text{Re}\ \{\frac{zG'(z)}{G(z)}\} > \beta$$

Further, since $G(z) \in S^*_\beta$, from theorem 1, we have

$$\text{Re}\ \{\frac{zg'(z)}{g(z)}\} > \beta \text{ for } |z| < r_o.$$

Let us compute

$$\frac{zF'(z)}{G(z)} = z^{1-c}[\frac{(z^cF(z))' - c\,z^{c-1}F(z)}{G(z)}]$$

$$= \frac{z^cf(z) - c\int_0^z f(t)\,t^{c-1}\,dt}{\int_0^z g(t)\,t^{c-1}\,dt}$$

Also for some $\omega(z)$ satisfying the conditions of lemma 1, we have

$$\frac{\frac{zF'(z)}{G(z)} - \lambda}{1-\lambda} = \frac{1-\omega(z)}{1+\omega(z)}$$

or, $$P(z) = \frac{zF'(z)}{G(z)} = 1 - \frac{(1-2\lambda)\omega(z)}{1+\omega(z)}$$

Thus we can write

$$P(z) = \frac{z^c f(z) - c \int_0^z t^{c-1} f(t)\, dt}{\int_0^z t^{c-1} g(t)\, dt}$$

$$z^c f(z) - c \int_0^z t^{c-1} f(t)\, dt = P(z) \int_0^z t^{c-1} g(t)\, dt \tag{2.2.10}$$

Differentiating (2.2.10) with respect to z, we obtain

$$z^c f'(z) + cz^{c-1} f(z) - cz^{c-1} f(z) = P(z) z^{c-1} g(z) + P'(z) \int_0^z t^{c-1} g(t)\, dt.$$

Hence,

$$\frac{zf'(z)}{g(z)} = P(z) + \frac{P'(z)}{g(z)} z^{1-c} \int_0^z t^{c-1} g(t)\, dt$$

$$\operatorname{Re} \left\{\frac{zf'(z)}{g(z)}\right\} = \operatorname{Re} \{P(z)\} + \operatorname{Re} \left\{\frac{z^{1-c} P'(z)}{g(z)} \int_0^z t^{c-1} g(t)\, dt\right\} \tag{2.2.11}$$

Now

$$\frac{1}{z^c g(z)} \int_0^z t^{c-1} g(t)\, dt = \frac{1}{z^c g(z)} \left[\frac{1}{c+1} \{z^c G(z)\}\right] \tag{2.2.12}$$

$$= \frac{G(z)}{z^{1-c}[cz^{c-1} G(z) + z^c G'(z)]}$$

$$= \frac{G(z)}{cG(z)+zG'(z)} .$$

Since $G(z) \in S^*_\beta$, we also have

(2.2.13) $$\frac{zG'(z)}{G(z)} = 1 - \frac{(1-2\beta)\ V(z)}{1+V(z)}$$

where $V(z)$ satisfies the conditions of lemma 1. Therefore we have from (2.2.13) that

(2.2.14) $$[c + \frac{zG'(z)}{G(z)}]^{-1} = [c + \frac{1-(1-2\beta)V(z)}{1+V(z)}]^{-1}$$

$$= [\frac{(c+1)+(c-1+2\beta)V(z)}{1+V(z)}]^{-1}$$

Hence we have from (2.2.12) and (2.2.14) that

(2.2.15) $$[\int_0^z t^{c-1}\ g(t)\ dt]/z^c g(z) = [\frac{(c+1)+(c-1+2\beta)V(z)}{1+V(z)}]^{-1}$$

Thus by using (2.2.15) we obtain from (2.2.11) that

(2.2.16) $$\mathrm{Re}\{\frac{zf'(z)}{g(z)}\}-\lambda \geq \mathrm{Re}\{P(z)-\lambda\}- \left|\frac{(c+1)+(c-1+2\beta)V(z)}{1+V(z)}\right|^{-1} |z||P'(z)|$$

$$\geq \mathrm{Re}\ \{P(z)\ -\lambda\} - [\frac{1+|z|}{(c+1)+(c-1+2\beta)|z|}][\frac{2|z|\ \mathrm{Re}\{P(z)\ -\lambda\}}{(1-|z|^2)}]$$

$$= \mathrm{Re}\{P(z)\ -\ \lambda\}[\frac{(c+1)-|z|(-c+1-2\beta+c+1+2)\ -\ (c+2\beta-1)|z|^2}{[(c+1)+(c-1+2\beta)|z|][1-|z|]}]$$

$$= \operatorname{Re}\{P(z) - \lambda\}\left[\frac{(c+1)-2(2-\beta)|z| - (c+2\beta-1)|z|^2}{(1-|z|)[(c+1)+(c+2\beta-1)|z|]}\right]$$

The inequality (2.2.16) implies that $f(z) \in \Gamma(\lambda,\beta)$ with respect to the function $g(z)$ if the value of r is smaller than the root of the polynomial

$$(c+1) - 2(2-\beta)\ |z| - (c+2\beta-1)\ |z|^2 = 0$$

i.e., $$r_o = |z| = \frac{-(2-\beta) + \sqrt{3+\beta^2+c^2+2\beta c-2\beta}}{(c+2\beta-1)}$$

This completes the proof of theorem 3. Sharpness of the theorem follows from theorem 2.

As a corollary to this theorem, theorem [H] of Bernardi follows by taking $\beta = 0$.

Theorem 4 : Let $F(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular and have the property $\operatorname{Re}\{F'(z)\} > \beta$ for $|z| < 1$, $f(z) = \frac{1}{1+c} z^{1-c}[z^c F(z)]'$, $c = 1,2,3,\ldots$. Then $\operatorname{Re}\{f'(z)\} > \beta$ for $|z| < r_1 = [-1 + \frac{(2+2c+c^2)^{1/2}}{1+c}]$. The result is sharp.

Proof : The proof given by Bernardi for theorem [J] remains valid except for the following change.

$$(1+c)\ \operatorname{Re}\{f'(z)\} \geq \operatorname{Re}\{P(z)\}\ \left[(1+c) - \frac{2|z|(\operatorname{Re}\{P(z)-\beta\})}{1-|z|^2}\right]$$

or $$(1+c)\operatorname{Re}\{f'(z)-\beta\} \geq \operatorname{Re}\{P(z)-\beta\}\left[(1+c) - \frac{2|z|}{1-|z|^2}\right]$$

This result is sharp as is seen by the example,

$$F(z) = (2\beta-1)z - 2(1-\beta)\ \log\ (1-z).$$

2.3. In this section we prove few theorems on the radius of convexity for the class $\bar{S}^*_\delta$ introduced by D.J. Wright [67] which forms a subclass of S^*_β.

Theorem 5. If $f(z) \in \bar{S}^*_\delta$ then $f(z)$ is convex for

$$|z| < \frac{(3-2\delta) - \sqrt{5-8\delta+4\delta^2}}{2(1-\delta)}$$

Proof : By lemma 4, we have

$$1 + \frac{zf''(z)}{f'(z)} = 1 + (1-\delta)z\phi(z) + \frac{z(1-\delta)[z\phi'(z)+\phi(z)]}{1+(1-\delta)z\phi(z)}$$

From above equation we have

$$\text{(2.3.1)} \qquad \mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} \geq 1 - (1-\delta)|z| - \frac{(1-\delta)|z|(|z||\phi'(z)|+|\phi(z)|)}{1-(1-\delta)|z||\phi(z)|}$$

But,

$$\text{(2.3.2)} \qquad |z\phi'(z) + \phi(z)| \leq |z||\phi'(z)|+|\phi(z)|$$

$$\leq \frac{|z|(1-|\phi(z)|^2)}{1-|z|^2} + |\phi(z)|$$

$$= \frac{(1-|z||\phi(z)|)(|z|+|\phi(z)|)}{(1-|z|^2)}$$

From (2.3.1) and (2.3.2) we obtain that

$$\text{(2.3.3)} \qquad \mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} \geq 1 - (1-\delta)|z| - \frac{(1-\delta)|z|(|z|+|\phi(z)|)}{(1-|z|^2}$$

The inequality (2.3.3) is obtained because of the fact that

$$\frac{1-|z||\phi(z)|}{1-(1-\delta)|z||\phi(z)|} - 1 = \frac{1-|z||\phi(z)|-1+|z||\phi(z)|-\delta|z||\phi(z)|}{1-(1-\delta)|z||\phi(z)|} \leq 0$$

Thus,

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq 1 - (1-\delta)|z| - \frac{(1-\delta)|z|(|z|+|\phi(z)|)}{1 - |z|^2}$$

$$\geq \frac{1-|z| - (1-\delta)|z|(1-|z|)-(1-\delta)|z|}{(1 -|z|)}$$

$$= \frac{1 - (3-2\delta)|z| + (1-\delta)|z|^2}{(1-|z|)}$$

If we write $|z| = r$ and

$$P(r) = 1 - (3 - 2\delta)\ r + (1 - \delta)r^2$$

then,

$$P'(r) = -(3- 2\delta) + 2(1-\delta)\ r$$

$$= -1 -(2-2\delta)\ (1-r) < 0$$

This shows that $P(r)$ is a decreasing function of r and further $P(0) = 1 > 0$. Hence we conclude that $P(r)$ will remain positive for all values of r which are smaller than the smallest positive root of $P(r) = 0$. But $P(r) = 0$ implies that

$$r = \frac{(3-2\delta)\pm\sqrt{9+4\delta^2-12\delta-4+4\delta}}{2(1-\delta)}$$

$$= \frac{(3-2\delta)\pm\sqrt{5-8\delta+4\delta^2}}{2(1-\delta)}$$

and the quantity under the radical sign is positive since

$$5-8\delta+4\delta^2 = 1+4(1-2\delta+\delta^2) = 1+4(1-\delta)^2 > 0$$

Hence the smallest positive root is

$$r = \frac{(3-2\delta) - \sqrt{5-8\delta+4\delta^2}}{2(1-\delta)}$$

since $(9+4\delta^2-12\delta) - (5-8\delta+\delta^2) = 4(1-\delta) > 0$. Hence every $f(z) \varepsilon \overline{S}^*_\delta$ maps $\{|z| < \frac{(3-2\delta)-\sqrt{5-8\delta+4\delta^2}}{2(1-\delta)}\}$ onto a convex domain. This completes the proof of the theorem.

As a corollary, we obtain the following result of D.J. Wright [67] (which is also due to R. Singh [60] by putting $\delta = 0$ in theorem 5.

Theorem ([Singh], [Wright]). *If* $f(z) \varepsilon \overline{S}^*_0$, *then* $f(z)$ *is convex for*

$$|z| < \frac{3-\sqrt{5}}{2}.$$

In particular, we obtain the following sharp result for functions belonging to the class $\overline{S}^*_\delta$, when $0 \le \delta \le \sqrt{5} - 2$.

Theorem 6. *If* $f(z) \varepsilon \overline{S}^*_\delta$ *with* $0 \le \delta \le \sqrt{5} - 2$, *then* $f(z)$ *is convex for*

$$|z| < \frac{3-\sqrt{5}}{2(1-\delta)}$$

and the result is sharp.

Proof : As in theorem 6, we have

$$\mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\} \geq 1 - (1-\delta)|z\phi(z)| - \frac{(1-\delta)|z|(1-|z\phi(z)|)(|z|+|\phi(z)|)}{(1-|z|^2)(1-(1-\delta)|z\phi(z)|)}$$

If we write $|z| = r, |\phi(z)| = x$ and

$$P(x) = [1-(1-\delta)r\,x]^2(1-r^2) - (1-\delta)r(1-rx)(r+x)$$

i.e., $P(x) = (1-r^2)[1-2(1-\delta)rx+(1-\delta)^2r^2x^2] - (1-\delta)r[r+x-r^2x-rx^2]$

$= x^2[(1-\delta)^2r^2(1-r^2)+(1-\delta)r^2]+x[-2(1-\delta)r(1-r^2)-(1-\delta)r+r^3(1-\delta)]+[(1-r^2)-(1-\delta)r^2]$

then,

$$P'(x) = 2x[(1-\delta)r^2\{(1-r^2)(1-\delta)+1\}] + (1-\delta)r[-3(1-r^2)]$$

and $P'(x)$ is negative if

$$2x[(1-r^2)+1]\ (1-\delta)r^2 \leq 3(1-\delta)r(1-r^2)$$

The above inequality will hold if

$$2(2-r^2)r \leq 3(1-r^2)$$

i.e., if

$$u(r) = 3 - 4r - 3r^2 + 2r^3 \geq 0$$

Obviously $u(r)$ is a decreasing function of r and minimum value of r, for which $u(r) \geq 0$, lies in the range $\frac{1}{2} < r < \frac{3}{4}$ for

$$u(\tfrac{1}{2}) = 3-2-\frac{3}{4}+\frac{1}{4} > 0 \quad \text{and} \quad u(\tfrac{3}{4}) = -\frac{27}{15}+\frac{54}{64} < 0$$

Choose in particular $r = \frac{1}{2}$, then $u(r) > 0$ for all $r \leq \frac{1}{2}$. For this choice of $r, P(x)$ is a decreasing function of x, hence

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$$

if $\min P(x) > 0$. Thus $f(z)$ will be convex if

$$(1-r^2)\,[1-(1-\delta)r]^2 - (1-\delta)r(1-r)(1+r) > 0$$

i.e., if, $\quad 1 - 3(1-\delta)r + (1-\delta)^2r^2 > 0$

Denote the left hand side of the above inequality by $Q(r)$. Then

$$Q'(r) = -3(1-\delta) + 2(1-\delta)^2\, r < 0$$

Hence $Q(r)$ is a decreasing function of r and so $Q(r) > 0$ if

$$r < \frac{3(1-\delta) - \sqrt{9(1-\delta)^2 - 4(1-\delta)^2}}{2(1-\delta)^2}$$

$$= \frac{(3 - \sqrt{5})}{2(1-\delta)}.$$

But we again observe that if $\delta \leq \sqrt{5} - 2$, then $\frac{3-\sqrt{5}}{2(1-\delta)} \leq \frac{1}{2}$.
This completes the proof of the first part of the theorem. In order to see that this result is sharp, let us consider the function

$$f(z) = z\, e^{(1-\delta)z} \text{ and } 0 \leq \delta \leq \sqrt{5} - 2.$$

Clearly

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = (1-\delta)|z| \leq (1-\delta)$$

and so $f(z) \in \overline{S}^*_\delta$ for every $0 \leq \delta \leq 1$. A further calculation shows that

$$\frac{zf''(z)}{f'(z)} + 1 = \frac{zf'(z)}{f(z)} + \frac{(1-\delta)z}{1+(1-\delta)z}$$

$$= 1 + (1-\delta)z + \frac{(1-\delta)z}{1+(1-\delta)z}$$

$$= \frac{1+3(1-\delta)z+(1-\delta)^2z^2}{1+(1-\delta)z}$$

Hence for $z = -\frac{(3-\sqrt{5})}{2(1-\delta)}$; $\frac{zf''(z)}{f'(z)} + 1 = 0$, so this implies that $f(z)$ is not convex in any disc of radius r for which $r > \frac{3-\sqrt{5}}{2(1-\delta)}$.

In course of the proof of this theorem; infact, we have proved the following :

<u>Theorem 7.</u> <u>If</u> $f(z) \in \overline{S}^*_\delta$ <u>then</u> $f(z)$ <u>is convex for</u>

$$|z| < \frac{3-\sqrt{5}}{2(1-\delta)}$$

<u>where</u> $0 \leq \delta \leq \frac{(2r_0-3)+\sqrt{5}}{2r_0}$ <u>and</u> r_0 <u>is the smallest positive root of the</u>

polynomial equation.

$$u(r) = 3-4r-3r^2+2r^3$$

and the result is sharp.

Remark : Our method of proof fails to give the exact radius of convexity for $f(z) \varepsilon \overline{S}^*_\delta$ when $r_o < \delta \leq 1$. However, the example $f(z) = ze^{(1-\delta)z}$ shows that result of theorem 7 is best possible for each δ.

2.4. If $W = f(z) = z + a_2z^2 + \sum_{n=3}^{\infty} a_n z^n \varepsilon S$, then in [20] it is shown that there is a number r_o such that $|z| < r_o$ is mapped onto a convex region and $r_o \geq 2 - \sqrt{3}$. The lower bound is attained for the class S^*_o.

In this section we investigate how the second coefficient in the expansion of $W = f(z) = z + a_2z^2 + \sum_{n=3}^{\infty} a_n z^n$ affects the radius of convexity of the function $W = f(z) = z + a_2z^2 + \sum_{n=3}^{\infty} a_nz^n \varepsilon S^*_\beta$. The corresponding problem for the class S^*_o is solved by Tepper which follows as a particular case from our theorem by taking $\beta = 0$. We further study some of the special cases when $\beta = 0$, $a_2 = 0$ or $\beta = \frac{1}{2}$. These classes possess some special properties and have also been studied by Strodacker [63], Schild [59], Gronwall [19] and others.

Specially, it is interesting to see how the n^{th} coefficient affects the radius of convexity for $f(z) \varepsilon S^*_\beta$.

Theorem 8. If $W = f(z) = z + a_2z^2 + \sum_{n=3}^{\infty} a_nz^n \varepsilon S^*_\beta$ then the radius of convexity of $W = f(z)$ of order β is given by the smallest positive root of the following polynomial equation.

$$(1-\beta)^2-\beta(1-\beta)|a_2||z|-[(5+2\beta)(1-\beta)^2+|a_2|^2]|z|^2-2[(1-\beta)|a_2|(4-\beta)]|z|^3 -$$

$$[(1-\beta)^2(5-4\beta)+|a_2|^2]|z|^4 - [(1-\beta)|a_2|\beta]|z|^5+(1-\beta)^2(1-2\beta)|z|^6 = 0.$$

For special cases the exact value of the root can be obtained.

Proof. Since $W = f(z) = z + a_2z^2 + \sum_{n=3}^{\infty} a_n z^n \in S^*_\beta$ therefore there exists a bounded regular function $\omega(z)$ satisfying the conditions of Schwarz's lemma, such that

$$\frac{zf'(z)}{f(z)} = \frac{1+(1-2\beta)\omega(z)}{1-\omega(z)} \tag{2.4.1}$$

A direct calculation yields that

$$\omega(z) = \frac{1}{2} a_2 \, z/(1-\beta) + \ldots$$

Write $\omega(z) = z\phi(z)$ in (2.4.1) and differentiating logarithmically we have

$$\frac{1}{z} + \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} = \frac{(1-2\beta)[z\phi'(z)+\phi(z)]}{1+(1-2\beta)z\phi(z)} + \frac{z\phi'(z)+\phi(z)}{1-z\phi(z)}$$

$$= \frac{2(1-\beta)[z\phi'(z)+\phi(z)]}{[1+(1-2\beta)z\phi(z)][1-z\phi(z)]} .$$

This implies that

$$\operatorname{Re}\{1-\beta+\frac{zf''(z)}{f'(z)}\} \geq \operatorname{Re}\{\frac{zf'(z)}{f(z)}-\beta\} - \left|\frac{2(1-\beta)z[z\phi'(z)+\phi(z)]}{[1-z\phi(z)][1+(1-2\beta)z\phi(z)]}\right| \tag{2.4.2}$$

From lemma 2 we have

$$|\phi'(z)| \leq \frac{1-|\phi(z)|^2}{(1-|z|^2)} \tag{2.4.3}$$

Hence we obtain that

$$|z\phi'(z)+\phi(z)| \le \frac{(|z|+|\phi(z)|)(1-|z||\phi(z)|)}{(1-|z|^2}$$

Thus

$$\left|\frac{z\phi'(z) + \phi(z)}{[1-z\phi(z)][1+(1-2\beta)z\phi(z)]}\right| \le \frac{(|z|+|\phi(z)|)\ (1-|z||\phi(z)|)}{(1-|z|^2)\,|1-z\phi(z)|\,|1+(1-2\beta)z\phi(z)|}$$

$$\le \frac{(|z|+|\phi(z)|)\,|1+z\phi(z)|}{(1-|z|^2)(1+|z||\phi(z)|)\,|1+(1-2\beta)z\phi(z)|}$$

Let us write $z\phi(z) = t\ e^{i\Phi}$ and investigate the value of Φ, for which the following expression attains its maximum.

$$h(\Phi) = \left|\frac{1+z\phi(z)}{1+(1-2\beta)z\phi(z)}\right|^2 = \frac{1+t^2+2t\cos\Phi}{1+t^2(1-2\beta)^2+2t(1-2\beta)\cos\Phi}.$$

Differentiating $h(\Phi)$ twice with respect to Φ we obtain

$$h'(\Phi) = \frac{-2t\sin\Phi\{1+t^2(1-2\beta)^2+2t(1-2\beta)\cos\Phi\}+2t(1-2\beta)\sin\Phi\{1+t^2+2t\cos\Phi\}}{\{1+t^2(1-2\beta)^2+2t\ (1-2\beta)\cos\Phi\}^2}$$

$$= \frac{-4\beta t\sin\Phi[(1-t^2)+2\beta t^2]}{1+t^2(1-2\beta)^2+2t(1-2\beta)\cos\Phi\}^2}$$

and

$$h''(\Phi) = \frac{-4\beta t\cos\Phi[(1-t^2)+2\beta t^2][1+t^2(1-2\beta)^2+2t(1-2\beta)\cos\Phi]^2+4\beta t\sin^2\Phi\ g(\Phi)}{\{1+t^2(1-2\beta)^2+2t(1-2\beta)\cos\Phi\}^4}$$

where

$$g(\Phi) = 2\,[(1-t)^2+2\beta t^2][1+t^2(1-2\beta)^2+2t(1-2\beta)\cos\Phi].\ 2t(1-2\beta)$$

From $h'(\Phi)$ and $h''(\Phi)$ it immediately follows that $h(\Phi)$ attains its maximum for $\Phi = 0$ on the circle $|z\phi(z)| = t < 1$. Hence

$$\left|\frac{1+z\phi(z)}{1+(1-2\beta)z\phi(z)}\right| \le \frac{1+|z|\,|\phi(z)|}{1+(1-2\beta)|z|\,|\phi(z)|}$$

Thus finally we have

$$(2.4.4)\quad \mathrm{Re}\{\frac{zf''(z)}{f'(z)}+1-\beta\} \geq \mathrm{Re}\{\frac{zf'(z)}{f(z)}-\beta\} - \frac{2(1-\beta)(|z|+|\phi(z)|)(1+|z|\,|\phi(z)|)}{(1-|z|^2)(1+|z|\,|\phi(z)|)(1+(1-2\beta)|z\phi(z)|)}$$

$$= \mathrm{Re}\,\{\frac{zf'(z)}{f(z)}-\beta\} - [\frac{2(1-\beta)(|z|+|\phi(z)|)}{(1-|z|^2)(1+(1-2\beta)|z|\,|\phi(z)|)}]$$

We wish to show that the expression in the square bracket on the right hand side of (2.4.4) is monotonic increasing with respect to $|\phi(z)|$. To do this consider

$$(2.4.5)\qquad g(x) = \frac{x+r}{(1+(1-2\beta)rx)},\quad 0 \le x = |\phi(z)| \le 1,\ 0 \le r = |z| < 1.$$

Differentiating (2.4.5), we have

$$g'(x) = \frac{1+(1-2\beta)rx - (1-2\beta)r(x+r)}{(1+(1-2\beta)rx)^2}$$

$$= \frac{(1-r^2)+2\beta r^2}{\{1+(1-2\beta)\,rx\}^2} > 0$$

Hence $g(x)$ is an increasing function with respect to $|\phi(z)|$. Hence by generalized Schwarz's lemma 2, we have

$$(2.4.6)\qquad |\phi(z)| \le \left|\frac{\omega(z)}{z}\right| \le \frac{2(1-\beta)|z|+|a_2|}{2(1-\beta)+|a_2||z|}$$

Applying (2.4.6) to (2.4.4) we have obtain

$$\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}+1-\beta\right\} \ge \frac{(1-\beta)(1-|z|^2\frac{2(1-\beta)|z|+|a_2|}{2(1-\beta)+|a_2||z|}}{1+|z|\frac{2(1-\beta)|z|+|a_2|}{2(1-\beta)+|a_2||z|}} - \frac{2(1-\beta)|z|[|z|+\frac{2|z|(1-\beta)+|a_2|}{2(1-\beta)+|a_2||z|}]}{(1-|z|^2)[1+\frac{(1-2\beta)|z|\{2|z|(1-\beta)+|a_2|\}}{2(1-\beta)+|a_2||z|}]}$$

$$= \frac{(1-\beta)^2(1-|z|^2)}{[(1-\beta)+|a_2||z|+(1-\beta)|z|^2]} - \frac{|z|[|a_2|+4(1-\beta)|z|+|a_2||z|^2]}{(1-|z|^2)[1+|a_2||z|+(1-2\beta)|z|^2]}$$

$$= \frac{N_\beta(|z|)}{D_\beta(|z|)}$$

where $N_\beta(|z|)$ and $D_\beta(|z|)$ are given by the following expressions.

$$\begin{aligned} N_\beta(|z|) &= (1-\beta)^2+|z|[(1-\beta)^2|a_2|-|a_2|(1-\beta)]+|z|^2[(1-\beta)^2(1-2\beta)-2(1-\beta)^2-|a_2|^2 - \\ &\quad 4(1-\beta)^2]+|z|^3[-2(1-\beta)^2|a_2|-|a_2|(1-\beta)-4(1-\beta)|a_2|-|a_2|(1-\beta)]+|z|^4[(1-\beta)^2 \\ &\quad -2(1-\beta)^2(1-2\beta)-|a_2|^2-4(1-\beta)^2]+|z|^5[(1-\beta)^2|a_2|-(1-\beta)|a_2|] \\ &\quad +|z|^6[(1-\beta)^2(1-2\beta)]. \\ &= (1-\beta)^2-\beta(1-\beta)|a_2||z|-[(5+2\beta)(1-\beta)^2+|a_2|^2]|z|^2 -2[(1-\beta)|a_2|. \\ &\quad (4-\beta)]|z|^3 - [(1-\beta)^2(5-4\beta)+|a_2|^2]|z|^4-[(1-\beta)|a_2|\beta]|z|^5 + \\ &\quad (1-\beta)^2(1-2\beta)|z|^6. \end{aligned}$$

and

$$D_\beta(|z|) = [(1-\beta)+|a_2||z|+(1-\beta)|z|^2][1-|z|^2][1+|a_2||z|+(1-2\beta)|z|^2].$$

Since we know that $w = f(z) \in S$ maps $|z| \le r$ onto a convex region of order β if and only if

$$\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}+1-\beta\right\} \geq 0 \quad , \quad (0 \leq \beta < 1),$$

therefore $W = f(z) = z + a_2z^2 + \sum_{n=3}^{\infty} a_nz^n$ will map $|z| \leq r$ onto a convex region if $N_\beta(|z|) \geq 0$ for $|z| \leq r$. Hence the radius of convexity of $W = f(z) = z + a_2z^2 + \sum_{n=3}^{\infty} a_nz^n$ is greater than or equal to the least positive root of $N_\beta(|z|) = 0$. This proves the theorem.

From the above theorem we deduce the following theorem of Tepper for the class S_0^* by taking $\beta = 0$.

<u>Theorem [Tepper].</u> <u>If $W = f(z) = z + a_2z^2 + \sum_{n=3}^{\infty} a_nz^n \in S_0^*$ and has the radius of convexity r_0, then</u>

$$r_0 \geq \frac{a_2 + \sqrt{a_2^2+32} - \sqrt{2a^2 + 2a\sqrt{a^2 + 32} + 16}}{4}$$

<u>where a_2 is real and $0 \leq a_2 \leq 2$.</u>

<u>Proof.</u> Putting $\beta = 0$ in the expression $N_\beta(|z|)$ and taking $|z| = r$, we have

$$N_0(r) = 1-[5+|a_2|^2]r^2 - 8|a_2|r^3 - [5+|a_2|^2]r^4 + r^6$$

$$= [1+|a_2|r+r^2][1-|a_2|r - 6r^2 - |a_2|r^3 + r^4]$$

Hence the radius of convexity for functions of S_0^* is greater than or equal to the smallest positive root of the equation

$$1 - |a_2|r - 6r^2 - |a_2|r^3 + r^4 = 0$$

and the theorem of Tepper follow.

Corollary 1. *If* $f(z) = z + \sum_{n=3}^{\infty} a_n z^n \in S^*_{\frac{1}{2}}$ *then the radius of convexity* r_o *for* $f(z)$ *is greater than or equal to* $\frac{\sqrt{-3+2\sqrt{3}}}{3}$.

Corollary 2. *If* $f(z) = z + \sum_{n=3}^{\infty} a_n z^n \in S^*_o$ *then the radius of convexity* r_o *is greater than or equal to* $\sqrt{2}-1$.

CHAPTER 3

Distortion Theorems For Univalent α-Spiral Functions

3.1 In this chapter we study the following classes of univalent functions:

Definition 1. Let $S(\alpha)$ denote the class of functions $f(z) = z+a_2z+\sum_{n=3}^{\infty} a_n z^n$ regular for $|z| < 1$ and having the additional property that

(3.1.1)
$$\operatorname{Re}\left\{\frac{e^{i\alpha} z f'(z)}{f(z)}\right\} \geq 0$$

for some real α such that $|\alpha| \leq \frac{\pi}{2}$.

Špaček [62] proved that, if $f(z) = z + a_2 z + \sum_{n=3}^{\infty} a_n z^n$ is regular for $|z| < 1$ and (3.1.1) is satisfied, then $f(z)$ is univalent for $|z| < 1$. Functions belonging to the class $S(\alpha)$ are called univalent α-spiral functions. In (3.1.1) equality is possible only for identity function $f(z) = z$, when $\alpha = \pm\frac{\pi}{2}$. Thus without loss of generality we may assume that $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ or else a strict inequality holds in (3.1.1).

Definition 2. If $f(z) \in S(\alpha)$, then $f(z)$ is said to be univalent α-spiral function of order ρ, if and only if,

(3.1.2)
$$\operatorname{Re}\left\{\frac{e^{i\alpha} z f'(z)}{f(z)}\right\} \geq \rho \ , \ z \in D$$

where $0 \leq \rho \leq 1$. We denote the class of functions $f(z) \in S(\alpha)$, and for which (3.1.2) holds, by $S_\rho(\alpha)$.

Clearly, $S_\rho(\alpha) \subseteq S(\alpha)$. We identify $S_0(\alpha)$ by $S(\alpha)$ and $S_\rho(0)$ by S^*_β, when $\rho = \beta$.

Definition 3. Let G denote the class of functions $P(z)$, which are regular for $|z| < 1$, whose real part is positive and which are normalized by the

condition $P(0) = 1$.

In this chapter we investigate how the second coefficient a_2 in $f(z)$ affects the upper and lower estimates for $f(z)$. These results are stated in the form of theorems . Some known results of Finkelstein [11], Tepper [65] and others follow as special cases from these theorems.

3.2. In this section we shall derive lower estimates of $|f(z)|$, $\mathrm{Re}\{\frac{ze^{i\alpha}f'(z)}{f(z)}\}$, and $|f'(z)|$.

Theorem .1. If $f(z) = z + a_2z^2 + \sum_{n=3}^{\infty} a_nz^n \in S_\rho(\alpha)$, then

$$(3.2.1.)\quad \mathrm{Re}\{\frac{ze^{i\alpha}f'(z)}{f(z)}\} \geq \frac{[2(1-\rho)(2\rho-1)\cos\alpha|z|^2+2|a_2|\rho|z|+2(1-\rho)\cos\alpha]\cos\alpha}{2(1-\rho)|z|^2\cos\alpha+2|a_2||z|+2(1-\rho)\cos\alpha}$$

Proof : Since $f(z) \in S_\rho(\alpha)$, therefore there exists a regular functinn $\omega(z)$ which satisfies the conditions of Schwarz's lemma and is such that

$$(3.2.2)\quad \frac{e^{i\alpha}zf'(z)\sec\alpha}{f(z)} - i\tan\alpha = \frac{1+(1-2\rho)\omega(z)}{1-\omega(z)} \in G.$$

From (3.2.2), we get by direct computation

$$\omega(z)[2(1-\rho)z+\sum_{n=2}^{\infty}\{n+i(n-1)\tan\alpha+(1-2\rho)\}a_nz^n = \sum_{n=2}^{\infty}(n-1)(1+i\tan\alpha)a_nz^n$$

Thus we obtain

$$\omega(z) = \frac{1}{2}a_2(\frac{1+i\tan\alpha}{1-\rho})z + \dots .$$

This implies that $\omega'(0) = \frac{1}{2}a_2(\frac{1+i\tan\alpha}{1-\rho})$. Hence, by using the generalized form of Schwarz's lemma, we get

(3.2.3) $$|\omega(z)| \le \frac{(|a_2|+2(1-\rho)|z| \cos \alpha)|z|}{2(1-\rho)\cos \alpha+|a_2||z|}$$

From (3.2.2) we have

$$\operatorname{Re}\left\{\frac{e^{i\alpha}zf'(z)\sec \alpha}{f(z)} - i \tan \alpha-\rho\right\} = \operatorname{Re}\left\{(1-\rho)\left\{\frac{1+\omega(z)}{1-\omega(z)}\right\}\right\}$$

$$= (1-\rho)\left\{\frac{1-|\omega(z)|^2}{|1-\omega(z)|^2}\right\}$$

$$\ge (1-\rho)\left[\frac{1-\dfrac{(2(1-\rho)|z|\cos \alpha+|a_2|)|z|}{2(1-\rho)\cos \alpha+|a_2||z|}}{1+\dfrac{(2(1-\rho)|z|\cos \alpha+|a_2|)|z|}{2(1-\rho)\cos \alpha+|a_2||z|}}\right]$$

Hence

$$\operatorname{Re}\left\{\frac{e^{i\alpha}zf'(z)}{f(z)}\right\} \ge \left[\frac{2(1-\rho)^2(1-|z|^2)\cos \alpha}{2(1-\rho)\cos \alpha+2|a_2||z|+2(1-\rho)|z|^2 \cos \alpha} + \rho\right] \cos \alpha$$

$$= \frac{[2(1-\rho)(2\rho-1)|z|^2 \cos \alpha+2|a_2|\rho|z|+2(1-\rho)\cos \alpha]\cos \alpha}{[2(1-\rho)|z|^2 \cos \alpha+2|a_2||z|+2(1-\rho)\cos \alpha]}.$$

This proves (3.2.1). The following example shows that the result of the theorem is sharp.

Example : Let

$$f(z) = z\left[\frac{(1-\rho)\cos \alpha}{[(1-\rho)\cos \alpha-a_2z+(1-\rho)z^2 \cos \alpha]}\right]^{(1-\rho)e^{-i\alpha}\cos \alpha}$$

where $0 \le a_2 \le 2(1-\rho)\cos \alpha$.

We need to verify that $f(z) \in S_\rho(\alpha)$

By differentiating $f(z)$ logarithmically with respect to z, we obtain

$$\frac{f'(z)}{f(z)} = \frac{1}{z} - \frac{e^{-i\alpha}(1-\rho)\cos\alpha[2(1-\rho)z\cos\alpha - a_2]}{(1-\rho)\cos\alpha - a_2 z + (1-\rho)\cos\alpha z^2}$$

Thus we get

$$\{ze^{i\alpha}\frac{f'(z)}{f(z)}\sec\alpha - i\tan\alpha - \rho\} = \frac{(1-\rho)^2(1-z^2)\cos\alpha}{(1-\rho)\cos\alpha - a_2 z + (1-\rho)z^2\cos\alpha}$$

This implies that

$$\mathrm{Re}\{ze^{i\alpha}\frac{f'(z)}{f(z)}\sec\alpha - i\tan\alpha - \rho\}/(1-\rho)^2\cos\alpha$$

$$= \mathrm{Re}\{\frac{(1-z^2)((1-\rho)\cos\alpha - a_2\bar{z} + (1-\rho)\bar{z}^2\cos\alpha)}{|(1-\rho)\cos\alpha - a_2 z + (1-\rho)z^2\cos\alpha|^2}$$

$$= \frac{(1-|z|^2)((1-\rho)\cos\alpha(1+|z|^2) - a_2\,\mathrm{Re}\{z\})}{|(1-\rho)\cos\alpha - a_2 z + (1-\rho)z^2\cos\alpha|^2},$$

$$\geq \frac{(1-|z|^2)(1-\rho)\cos\alpha(1-2\,\mathrm{Re}\{z\} + |z|^2)}{|(1-\rho)\cos\alpha - a_2 z + (1-\rho)\cos\alpha z^2|^2} \geq 0$$

This verifies that $f(z) \in S_\rho(\alpha)$. We further obtain

$$(3.2.4) \qquad (e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha)\sec\alpha = \frac{(1-\rho)\cos\alpha - \rho a_2 z + (1-\rho)(2\rho-1)z^2\cos\alpha}{(1-\rho)\cos\alpha - a_2 z + (1-\rho)z^2\cos\alpha}$$

Clearly (3.2.4) implies equality in (3.2.1) for z real and negative. This establishes the sharpness of the theorem 1 and so the proof is complete.

From theorem 1, we obtain the following result of Finkelstein [11] as a corollary by taking $\alpha = 0 = \rho$ in (3.2.1).

<u>Theorem</u> [Finkelstein]. <u>If</u> $f(z) = z + a_2 z^2 + \sum_{n=3}^{\infty} a_n z^n \in S^*(0)$, then

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \frac{(1-|z|^2)}{1+|a_2||z|+|z|^2}$$

<u>Theorem 2.</u> <u>If</u> $f(z) = z + a_2 z + \sum_{n=3}^{\infty} a_n z^n \in S_\rho(\alpha)$, then

$$|f(z)| \geq |z| \left\{\frac{(1-\rho)\cos\alpha}{(1-\rho)r^2 \cos\alpha + |a_2|r + (1-\rho)\cos\alpha}\right\}^{1-\rho} \left\{\left(\frac{1-r}{1+r}\right)\right\}^{\tan\alpha}$$

where $|z| = r$.

<u>Proof</u> : If $z = re^{i\theta}$, then from (3.2.2) we have

$$\operatorname{Re}\left\{re^{i\alpha}\frac{\partial}{\partial r}\log\left\{\frac{f(re^{i\theta})}{re^{i\theta}}\right\}\right\} = \operatorname{Re}\left\{\frac{e^{i\alpha}zf'(z)}{f(z)}\right\} - \cos\alpha$$

$$\geq \frac{\cos\alpha[2(1-\rho)(2\rho-1)r^2 \cos\alpha + 2r\rho|a_2| + 2(1-\rho)\cos\alpha]}{2(1-\rho)r^2 \cos\alpha + 2|a_2|r + 2(1-\rho)\cos\alpha} - \cos\alpha$$

or,
$$r\left\{\frac{\partial}{\partial r}\log\left|\frac{f(re^{i\theta})}{re^{i\theta}}\right|\right\}\cos\alpha - r\frac{\partial}{\partial r}\left\{\arg\left(\frac{f(z)}{z}\right)\right\}\sin\alpha$$

$$\geq \cos\alpha\left[\frac{2(1-\rho)(2\rho-1)r^2\cos\alpha + 2|a_2|\rho r + 2(1-\rho)\cos\alpha}{2(1-\rho)r^2 \cos\alpha + 2|a_2|r + 2(1-\rho)\cos\alpha} - 1\right]$$

$$= \frac{-2(1-\rho)r \cos\alpha[|a_2| + 2(1-\rho)r \cos\alpha]}{2(1-\rho)r^2\cos\alpha + 2|a_2|r + 2(1-\rho)\cos\alpha}$$

Thus

$$\frac{\partial}{\partial r}\log\left|\frac{f(re^{i\theta})}{r}\right| - \frac{\partial}{\partial r}\left\{\arg\left(\frac{f(z)}{z}\right)\right\}\tan\alpha \geq \frac{-(1-\rho)[|a_2| + 2(1-\rho)r \cos\alpha]}{(1-\rho)r^2\cos\alpha + |a_2|r + (1-\rho)\cos\alpha} \tag{3.2.5}$$

Integrating (3.2.5) between the limits $r = 0$ and $|z| = r < 1$, we obtain

$$\log \left|\frac{f(re^{i\theta})}{r}\right| - \tan\alpha \int_0^r \frac{\partial}{\partial r}\{\arg(\frac{f(re^{i\theta})}{re^{i\theta}})\}dr$$

$$\geq -(1-\rho)\log\left[\frac{(1-\rho)r^2\cos\alpha+|a_2|r+(1-\rho)\cos\alpha}{(1-\rho)\cos\alpha}\right]$$

Taking the exponents, we get

$$(3.2.6) \qquad \left|\frac{f(re^{i\theta})}{r}\right| \geq \left[\frac{(1-\rho)\cos\alpha}{(1-\rho)\cos\alpha+|a_2|r+(1-\rho)r^2\cos\alpha}\right]^{(1-\rho)} e^{Q(r,\theta)\tan\alpha}$$

where $Q(r,\theta) = \int_0^r \frac{\partial}{\partial r}\{\arg(\frac{f(z)}{z})\}\,dr$.

Now we shall compute the quantity $Q(r,\theta)$.

$$Q(r,\theta) = \int_0^r \frac{\partial}{\partial r}\arg\{\frac{f(z)}{z}\}\,dr = [\arg(\frac{f(re^{i\theta})}{re^{i\theta}})]_0^r$$

$$= \arg(\frac{f(re^{i\theta})}{re^{i\theta}}) - \arg 1 = \arg(\frac{f(re^{i\theta})}{re^{i\theta}})$$

But it is wellknown that ([20],p.139)

$$(3.2.7) \qquad -\log(\frac{1+r}{1-r}) \leq \arg(\frac{f(re^{i\theta})}{re^{i\theta}}) \leq \log(\frac{1+r}{1-r})$$

Hence from (3.2.6) and (3.2.7) we obtain finally that

$$\left|\frac{f(re^{i\theta})}{r}\right| \geq \left[\frac{(1-\rho)\cos\alpha}{(1-\rho)r^2\cos\alpha+|a_2|r+(1-\rho)\cos\alpha}\right]^{1-\rho}\exp[-\tan\alpha\log(\frac{1+r}{1-r})]$$

which proves the theorem.

From theorem 2, we obtain the following result of Tepper [65] which in turn includes a result of Finkelstien [11] by taking $\alpha = 0$.

Theorem [Tepper]. *If* $f(z) = z + a_2 z^2 + \sum_{n=3}^{\infty} a_n z^n \in S^*_\rho$, then

$$|f(z)| \geq |z| \left[\frac{(1-\rho)}{(1-\rho)|z|^2 + |a_2||z| + (1-\rho)} \right]^{(1-\rho)}$$

Theorem 3. *If* $f(z) = z + a_2 z^2 + \sum_{n=3}^{\infty} a_n z^n$ belongs to the class $S_\rho(\alpha)$, then

$$(3.2.8)\quad |f'(z)| \geq \left(\frac{1-r}{1+r}\right)^{\tan\alpha} \left[\frac{\{2(1-\rho)\cos\alpha\}^{1-\rho}\cos\alpha[(1-\rho)(2\rho-1)r^2\cos\alpha + \rho|a_2|r + (1-\rho)\cos\alpha]}{\{(1-\rho)r^2\cos\alpha + |a_2|r + (1-\rho)\cos\alpha\}^{2-\rho}} \right.$$

Proof : From theorem 1, we have

$$(3.2.9)\qquad \left|\frac{zf'(z)}{f(z)}\right| \geq \operatorname{Re}\left\{\frac{e^{i\alpha} zf'(z)}{f(z)}\right\}$$

$$\geq \frac{[2(1-\rho)(2\rho-1)r^2\cos\alpha + 2|a_2|\rho r + 2(1-\rho)\cos\alpha]\cos\alpha}{[2(1-\rho)r^2\cos\alpha + 2|a_2|r + 2(1-\rho)\cos\alpha]}$$

while from theorem 2, we have

$$(3.2.10)\quad \left|\frac{f(z)}{z}\right| \geq \left[\frac{2(1-\rho)\cos\alpha}{2(1-\rho)r^2\cos\alpha + 2|a_2|r + 2(1-\rho)\cos\alpha}\right]^{1-\rho} \exp\left[-\tan\alpha \log\left(\frac{1+r}{1-r}\right)\right]$$

Using (3.2.9) and (3.2.10) together, we get

$$|f'(z)| = \left|\frac{zf'(z)}{f(z)}\right| \left|\frac{f(z)}{z}\right|$$

$$\geq \frac{\{2(1-\rho)\cos\alpha\}^{1-\rho}[2(1-\rho)(2\rho-1)r^2\cos\alpha + 2|a_2|r\rho + 2(1-\rho)\cos\alpha]\cos\alpha}{(2(1-\rho)r^2\cos\alpha + 2|a_2|r + 2(1-\rho)\cos\alpha)^{2-\rho}} \left(\frac{1-r}{1+r}\right)^{\tan\alpha}$$

This completes the proof of theorem 3.

If we take $\alpha = 0$ in (3.2.8) we get the following result of Tepper which includes in particular a result of Finkelstien for $\rho = 0$

Theorem [Tepper]. If $f(z) = z + a_2 z^2 + \sum_{n=3}^{\infty} a_n z^n$ belongs to the class $S_\rho(0)$ i.e., to S^*_ρ , then

$$(3.2.11) \quad |f'(z)| \geq \frac{\{2(1-\rho)\}^{1-\rho}[(1-\rho)+|a_2|r\rho+(1-\rho)(2\rho-1)r^2]}{[(1-\rho)r^2+|a_2|r+(1-\rho)]^{2-\rho}}$$

3.3 In this section we shall obtain some upper estimates of $|f(z)|$, $\mathrm{Re}\,\{\frac{e^{i\alpha}zf'(z)}{f(z)}\}, |f'(z)|$.

Theorem 4. If $f(z) = z + a_2 z^2 + \ldots \in S_\rho(\alpha)$ then

$$(3.3.1) \qquad \mathrm{Re}\,\{\frac{e^{i\alpha}zf'(z)}{f(z)}\} \leq \frac{\cos\alpha+|a_2||z|+\cos\alpha\,|z|^2(1-2\rho)}{(1-|z|^2)}$$

Proof : As in the proof of theorem 1, we have

$$\frac{zf'(z)e^{i\alpha}\sec\alpha}{f(z)} - i\tan\alpha = \frac{1+(1-2\rho)\omega(z)}{1-\omega(z)}$$

and $\omega(z) = \frac{1}{2}a_2(\frac{1+i\tan\alpha}{1-\rho})z + \ldots$.

Hence we have

$$\mathrm{Re}\,\{\frac{e^{i\alpha}zf'(z)}{f(z)}\sec\alpha-\rho\} = (1-\rho)\,\mathrm{Re}\,\{\frac{1+\omega(z)}{1-\omega(z)}\} \leq (1-\rho)\,\left|\frac{1+\omega(z)}{1-\omega(z)}\right|$$

$$\leq \frac{(1-\rho)|z|^2\cos\alpha+|a_2||z|+(1-\rho)\cos\alpha}{(1-|z|^2)\cos\alpha}$$

$$\mathrm{Re}\,\{\frac{e^{i\alpha}zf'(z)}{f(z)}\} \leq \frac{[(1-2\rho)\cos\alpha|z|^2+|a_2||z|+\cos\alpha]}{(1-|z|^2)}$$

This establishes (3.3.1). The following example shows that the theorem is sharp.

Example : Let

$$f(z) = z\left[\frac{1}{1-z^2}\right]^{(1-\rho)e^{-i\alpha}\cos\alpha}\left[\frac{1-z}{1+z}\right]^{\frac{1}{2}a_2e^{-i\alpha}}$$

where $0 \leq a_2 \leq 2(1-\rho)\cos\alpha$.

Differentiating $f(z)$ logarithmically with respect to z, we obtain

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \frac{[-a_2e^{-i\alpha}+2e^{-i\alpha}z(1-\rho)\cos\alpha]}{(1-z^2)}$$

This gives us

$$\frac{e^{i\alpha}zf'(z)}{f(z)} - i\sin\alpha = \frac{(1-2\rho)\cos\alpha\ z^2 - a_2 z + \cos\alpha}{(1-z^2)}$$

This verifies that equality in (3.3.1) holds for real negative z. Now we shall prove the main part that $f(z)$ is a member of $S_\rho(\alpha)$. From above we have

$$\frac{e^{i\alpha}zf'(z)\sec\alpha}{f(z)} - i\tan\alpha - \rho = \frac{z^2[1-2\rho+\rho]\cos\alpha - a_2z + (1-\rho)\cos\alpha}{(1-z^2)\cos\alpha}$$

$$= \frac{(1-\rho)\cos\alpha\ z^2 - a_2 z + (1-\rho)\cos\alpha}{(1-z^2)\cos\alpha}$$

Thus,

$$\mathrm{Re}\left\{\frac{e^{i\alpha}zf'(z)\sec\alpha}{f(z)} - i\tan\alpha - \rho\right\} \geq \frac{(1-|z|^2)(1-\rho)\cos\alpha(1-2\,\mathrm{Re}\{z\}+|z|^2)}{|1-z^2|^2\cos\alpha}$$

$$\geq \frac{(1-|z|^2)(1-\rho)\cos\alpha(1-|z|)^2}{|1-z^2|^2\cos\alpha} \geq 0$$

Hence $f(z) \in S_\rho(\alpha)$.

Theorem 5. If $f(z) = z + a_2z^2 + \ldots$ belongs to the class $S_\rho(\alpha)$, then

$$|f(z)| \le |z| \left[\left(\frac{1+|z|}{1-|z|}\right)^{\frac{1}{2}|a_2|\sec\alpha+\tan\alpha} \left(\frac{1}{1-|z|^2}\right)^{(1-\rho)}\right]$$

Proof : Taking $z = re^{i\theta}$, and using (3.3.1) we get

$$r\left[\cos\alpha \frac{\partial}{\partial r}\left\{\log\left|\frac{f(re^{i\theta})}{re^{i\theta}}\right|\right\} - \frac{\partial}{\partial r}\left\{\arg\left(\frac{f(re^{i\theta})}{re^{i\theta}}\right)\right\}\sin\alpha\right] = \left[\mathrm{Re}\left\{\frac{e^{i\alpha}zf'(z)}{f(z)}\right\} - \cos\alpha\right]$$

$$\le \frac{r^2(1-2\rho)\cos\alpha+|a_2|r+\cos\alpha}{(1-r^2)} - \cos\alpha$$

$$= \frac{r(|a_2|+2(1-\rho)r\cos\alpha)}{(1-r^2)}$$

Hence, we have

$$\frac{\partial}{\partial r}\log\left|\frac{f(re^{i\theta})}{re^{i\theta}}\right| - \frac{\partial}{\partial r}\left(\arg\left\{\frac{f(re^{i\theta})}{re^{i\theta}}\right\}\right)\tan\alpha \le \frac{(|a_2|+2(1-\rho)\cos\alpha\, r)\sec\alpha}{(1-r)(1+r)}$$

Let us write

$$|a_2|\sec\alpha+2(1-\rho)r = (A+B) + (A-B)r$$

where

(3.3.2) $$A = \frac{1}{2}[|a_2|+2(1-\rho)\cos\alpha]\sec\alpha$$

and

(3.3.3) $$B = \frac{1}{2}[|a_2|-2(1-\rho)\cos\alpha]\sec\alpha$$

Thus, we have

$$(3.3.4)\quad \frac{\partial}{\partial r}\{\log\left|\frac{f(re^{i\theta})}{r}\right|\}-\frac{\partial}{\partial r}\,(\arg\{\frac{f(re^{i\theta})}{re^{i\theta}}\})\tan\alpha \le \frac{A}{1-r}+\frac{B}{1+r}$$

Integrating (3.3.4) from 0 to $r < 1$, we obtain

$$\log\left|\frac{f(re^{i\theta})}{r}\right| - (\arg\{\frac{f(re^{i\theta})}{re^{i\theta}}\})\tan\alpha \le \log\{(1+r)^B/(1-r)^A\}$$

Taking the exponents, we get

$$|f(re^{i\theta})| \le r\left[\frac{(1+r)^B}{(1-r)^A}\right]\exp\left[\tan\alpha\,\arg\{\frac{f(re^{i\theta})}{re^{i\theta}}\}\right]$$

From (3.2.7) we have

$$|f(re^{i\theta})| \le r\,\frac{(1+r)^{B+\tan\alpha}}{(1-r)^{A+\tan\alpha}}$$

Now substituting the values of A and B from (3.3.2) and (3.3.3) we have

$$|f(re^{i\theta})| \le r\left(\frac{1+r}{1-r}\right)^{\frac{1}{2}|a_2|\sec\alpha+\tan\alpha}\left(\frac{1}{1-r^2}\right)^{(1-\rho)}$$

This completes the proof of theorem 5.

As a corollary to theorem 5, we get the following theorem of Tepper [65] which in turn includes a result of Finkelstein [11] for $\rho = 0$.

Theorem [Tepper] : If $W = f(z) = z + a_2z^2 + \sum_{n=3}^{\infty} a_n z^n \in S_\rho(0)$, i.e., starlike function of order ρ, then

$$|f(z)| \le |z|\left[\frac{1+|z|}{1-|z|}\right]^{\frac{1}{2}|a_2|}\frac{1}{(1-|z|^2)^{1-\rho}}$$

Theorem [Finkelstein] : If $W = f(z) = z + a_2 z^2 + \sum_{n=3}^{\infty} a_n z^n \in S_o(0)$ i.e. starlike function, then

$$|f(z)| \leq |z| \left[\frac{1+z}{1-z} \right]^{\frac{1}{2}|a_2|} \frac{1}{(1-|z|^2)}$$

Theorem 6. If $W = f(z) = z + a_2 z^2 + \sum_{n=3}^{\infty} a_n z^n$ belongs to the class $S_\rho(\alpha)$ then

$$|f'(z)| \leq \left[\frac{2(1-\rho)\cos\alpha + (1+|1-2\rho|)|a_2 z| + 2(1-\rho)|1-2\rho||z|^2 \cos\alpha}{2(1-\rho)(1-|z|^2)} + |\sin\alpha|\right] \times$$

$$\times \left[\left(\frac{1}{1-|z|^2}\right)^{1-\rho} \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}|a_2|\sec\alpha + \tan\alpha}\right]$$

Proof : Since $P(z) \in G$, we have

$$|P(z)| = \left|\frac{e^{i\alpha} z f'(z) \sec\alpha}{f(z)} - \tan\alpha\right| \leq \frac{1+|(1-2\rho)||\omega(z)|}{1-|\omega(z)|}$$

$$= \frac{2(1-\rho)\cos\alpha + (1+|1-2\rho|)|a_2||z| + 2|1-2\rho|(1-\rho)|z|^2\cos\alpha}{2(1-\rho)(1-|z|^2)\cos\alpha}$$

Hence

$$(3.3.5) \quad \left|\frac{zf'(z)}{f(z)}\right| \leq \left[\frac{2(1-\rho)\cos\alpha + (1+|1-2\rho|)|a_2||z| + 2(1-\rho)|1-2\rho||z|^2\cos\alpha}{2(1-\rho)(1-|z|^2)\cos\alpha} + |\tan\alpha|\right]\cos\alpha$$

By using (3.3.5) and theorem 5, the result of the theorem follows.

Incidently, we have also proved a bound for $\left|\frac{zf'(z)}{f(z)}\right|$ which we state as a corollary as follows.

Corollary. If $f(z) = z + a_2z^2 + \sum_{n=3}^{\infty} a_nz^n$ belongs to the class $S_\rho(\alpha)$ then

$$\left|\frac{zf'(z)}{f(z)}\right| \leq \left[\frac{2(1-\rho)\cos\alpha+(1+|1-2\rho|)|a_2||z|+2(1-\rho)|1-2\rho||z|^2\cos\alpha}{2(1-\rho)(1-|z|^2)} + |\sin\alpha|\right]$$

From corollary, we deduce the following theorem of Finkelstein by taking $\rho = 0 = \alpha$.

Theorem [Finkelstein]: If $f(z) = z + a_2z^2 + \sum_{n=3}^{\infty} a_n z^n \in S_o(0)$, then

$$\left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+|a_2||z|+|z|^2}{(1-|z|^2)}$$

CHAPTER 4

Coefficient Estimates and a Growth Theorem For Univalent α-Spiral Functions

4.1. In this chapter we study the coefficient problem for univalent α-spiral functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$. We derive from theorem 1 of this chapter, the truth of Bieberbach's conjecture for the coefficients of normalized univalent α-spiral functions. Infact, we obtain $|a_n| \leq n \cos \alpha$ for all $n = 2,3,4,\ldots$, which gives a better estimate for the coefficients than that obtained by Špaček [62]. This evidently implies that if $\alpha=\pm\frac{\pi}{2}$, then the only normalized α-spiral function is the identity function $f(z) = z$. Further, the equality, $|a_n| = n$, $n > 1$, is not possible for any regular normalized univalent α-spiral function for which $\alpha \neq 0$. The proof of this theorem involves the use of Herglotz's theorem [1] and is more direct. Some coefficient estimates have recently been obtained by Libera [32] also, by using Clunies method [8]. Recently, MacGregor [36] has generalized a theorem of Golusin [15] for univalent starlike functions of the form $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$. He further remarks that the same theorem remains true for univalent α-spiral functions of the form $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$. Here we obtain a more refined coefficient estimate for this class of univalent α-spiral functions. We also discuss the growth of $f(z) = z + \sum_{n=1}^{\infty} a_n z^n \in S(\alpha)$. We conform to the same symbols as in chapter 3, and start by proving a few lemmas which we shall need.

Lemma 1. Let

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \; ; \; |z| < 1 \tag{4.1.1}$$

belong to the class $S(\alpha)$ and let F be defined by

(4.1.2) $$F(z) = \frac{ze^{i\alpha}f'(z)}{f(z)}$$

Then, for $0 \leq r < 1$

(4.1.3) $$\int_0^{2\pi} |f(re^{i\theta})|^2 \operatorname{Re}\{F(re^{i\theta})\}d\theta = 2\pi\, A_r(f) \cos\alpha$$

and

(4.1.4) $$\int_0^{2\pi} |f(re^{i\theta})|^2 \operatorname{Im}\{F(re^{i\theta})\}\, d\theta = 2\pi\, A_r(f) \sin\alpha$$

where $A_r(f)$ is the area enclosed by the image of $|z| \leq r$ by f.

Proof : Fix, $0 \leq r < 1$. Since f is univalent we have ([20],p.2)

(4.1.5) $$2\pi\, A_r(f) = 2\pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}$$

With $z = re^{i\theta}$, Parsevall's theorem shows that (4.1.5) is also equal to

(4.1.6) $$\int_0^{2\pi} [\overline{f(z)}\, zf'(z)]d\theta = \int_0^{2\pi} |f(z)|^2\, e^{-i\alpha}F(z)d\theta = 2\pi\, A_r(f)$$

Hence, from (4.1.5) and (4.1.6) we have

(4.1.7) $$2\pi\, A_r(f) = \int_0^{2\pi} [\cos\alpha \operatorname{Re}\{F(z)\}+\sin\alpha \operatorname{Im}\{F(z)\}]|f(z)|^2 d\theta$$

and

(4.1.8) $$0 = \int_0^{2\pi} [-\sin\alpha \operatorname{Re}\{F(z)\}+\cos\alpha \operatorname{Im}\{F(z)\}]|f(z)|^2\, d\theta$$

The equations (4.1.7) and (4.1.8) together imply the result of lemma 1.

Lemma 2. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n, |z| < 1$, belong to the class $S(\alpha)$ and $\{s_n\}$ be the sequence of complex numbers defined by

(4.1.9) $$s_n(t) = \sum_{k=1}^{n} a_k e^{ikt} \quad \text{for } n = 1,2,3,\ldots .$$

Then

(4.1.10) $$\int_0^{2\pi} |s_n(t)|^2 d\mu_f(t) = \sum_{k=1}^{n} k|a_k|^2 \cos\alpha$$

where $\mu_f(t)$ is a non-decreasing function such that

(4.1.11) $$\int_0^{2\pi} |d\mu_f(t)| \leq 1.$$

Proof : Let $F(z)$ be given by (4.1.2) and $\mu_f(t)$ satisfy (4.1.11). By Herglotz's theorem ([1], p.47), we have

(4.1.12) $$\operatorname{Re}\{F(z)\} = \int_0^{2\pi} \frac{(1-r^2)}{|1-ze^{-it}|^2} d\mu_f(t)$$

Hence for fixed r, $0 \leq r < 1$, we obtain with the help of lemma 1 and (4.1.12)

$$2\pi \sum_{n=1}^{\infty} r^{2n} \sum_{k=1}^{n} k|a_k|^2 = 2\pi \sum_{n=1}^{\infty} \frac{n|a_n|^2 r^{2n}}{(1-r^2)}$$

$$= \frac{2\pi}{1-r^2} \frac{\sec\alpha}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \operatorname{Re}\{F(re^{i\theta})\} d\theta$$

$$= \frac{\sec\alpha}{1-r^2} \int_0^{2\pi} |f(re^{i\theta})|^2 \left[\int_0^{2\pi} \frac{(1-r^2)}{|1-re^{i(\theta-t)}|^2} d\mu_f(t)\right] d\theta$$

$$= \sec\alpha \int_0^{2\pi} [\int_0^{2\pi} |\frac{f(re^{i\theta})}{1-re^{i(\theta-t)}}|^2 d\theta] d\mu_f(t)$$

$$= \sec\alpha \int_0^{2\pi} [\int_0^{2\pi} |\sum_{n=1}^{\infty} z^n e^{-int} s_n(t)|^2 d\theta] d\mu_f(t)$$

$$= 2\pi \sec\alpha \int_0^{2\pi} [\sum_{n=1}^{\infty} |s_n(t)|^2 r^{2n}] d\mu_f(t)$$

The desired result of lemma 2 follows on comparison of the coefficients of powers of r on both sides.

Lemma 3. We have

$$(1-\rho)[(1-\rho)+\sum_{m=1}^{q-1}(mk+1-\rho)\{\frac{1}{m!}\prod_{\mu=0}^{m-1}(\mu+\frac{2(1-\rho)\cos^2\alpha}{k})(1+(\frac{2(1-\rho)\sin\alpha\cos\alpha}{\mu k+2(1-\rho)\cos^2\alpha})^2)^{\frac{1}{2}}\}^2]\cos^2\alpha$$

$$= [\frac{k}{2(q-1)^2}\prod_{\mu=0}^{q-1}(\mu+\frac{2(1-\rho)\cos^2\alpha}{k})(1+(\frac{2(1-\rho)\cos\alpha\sin\alpha}{\mu q+2(1-\rho)\cos^2\alpha})^2)^{\frac{1}{2}}]^2$$

where $0 \leq \rho \leq 1$, α is real with $|\alpha| \leq \frac{\pi}{2}$ and $q = 2, 3, 4, \ldots$.

Proof : First of all we will verify the lemma for $q = 2$. Thus for $q = 2$, i.e., for $m = 1$, we have

$$4(1-\rho)\cos^2\alpha \quad [(1-\rho)+(k+1-\rho)\{\frac{2(1-\rho)\cos^2\alpha}{k} \frac{1}{\cos\alpha}\}^2]$$

$$= \frac{4(1-\rho)^2}{k^2} [(k+2(1-\rho)\cos^2\alpha)^2 + 4(1-\rho)^2\cos^2\alpha\sin^2\alpha]\cos^2\alpha$$

$$= \frac{4(1-\rho)^2}{k^2} [k+2(1-\rho)\cos^2\alpha]^2 [1+(\frac{2(1-\rho)\cos\alpha\sin\alpha}{k+2(1-\rho)\cos^2\alpha})^2]\cos^2\alpha$$

$$= [k \prod_{\mu=0}^{1} (\mu + \frac{2(1-\rho)\cos^2\alpha}{k})(1+(\frac{2(1-\rho)\cos\alpha\sin\alpha}{\mu k + 2(1-\rho)\cos^2\alpha})^2)^{\frac{1}{2}}]^2$$

This verifies the lemma for $q = 2$. Now assume that the lemma is true for $m = 1,2,3,\ldots,q-1$. Then

$$4(1-\rho)\cos^2\alpha[(1-\rho)+(\sum_{m=1}^{q-1} + \sum_{m=q}^{q})\{(mk+1-\rho)[\frac{1}{m!} \prod_{\mu=0}^{m-1} (\mu+\frac{2(1-\rho)\cos^2\alpha}{k})(1 + (\frac{2(1-\rho)\cos\alpha\sin\alpha}{\mu k+2(1-\rho)\cos^2\alpha})^2)^{\frac{1}{2}}]^2\}] = [4(1-\rho)\cos^2\alpha\, qk + k^2q^2 + 4(1-\rho)^2\cos^2\alpha] \times$$

$$[\frac{1}{q!} \prod_{\mu=0}^{q-1} (\mu+\frac{2(1-\rho)\cos^2\alpha}{k})(1+(\frac{2(1-\rho)\cos\alpha\sin\alpha}{\mu k+2(1-\rho)\cos^2\alpha})^2)^{\frac{1}{2}}]^2$$

$$= [\frac{k}{q!} \prod_{\mu=0}^{q} (\mu + \frac{2(1-\rho)\cos^2\alpha}{k})\{1 + (\frac{2(1-\rho)\cos\alpha\sin\alpha}{\mu k + 2(1-\rho)\cos^2\alpha})^2\}^{\frac{1}{2}}]^2$$

Thus if the lemma is true for $m = 1, 2, 3, \ldots, q-1$, it is also true for $m=q$. But we have already verified the truth of the lemma for $m = 1$. This completes the proof of the lemma.

Next we state a lemma due to Hayman ([21],p.40) and derive some of its consequences which we shall need.

Lemma 4 [Hayman]: Suppose that

(4.1.13) $$\psi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n = u + i v$$

is regular in the disc $|z| < 1$ and $u > 0$ there. Then the limit

(4.1.14) $$A(\theta) = \lim_{r \to 1} (\frac{1-r}{1+r})\psi(re^{i\theta})$$

exists. The set of distinct values $\theta = \theta_\nu$ in $0 \leq \theta < 2\pi$ for which

$\alpha_\nu = A(\theta_\nu) \neq 0$ is countable and $\alpha_\nu > 0$, $\Sigma\, \alpha_\nu \leq 1$.

Further, we have

$$(4.1.15) \qquad \lim_{r \to 1} (1-r) \sum_{n=1}^{\infty} |b_n|^2 r^{2n} = 2\Sigma \alpha_\nu^{\ 2}.$$

Infact , the above lemma implies that

$$(4.1.16) \qquad \lim_{r \to 1} (1-r) \sum_{n=1}^{\infty} |b_n|^2 r^n = 4\Sigma \alpha_\nu^{\ 2}$$

If instead of (4.1.13) we choose the function

$$F(z) = e^{i\alpha} + \sum_{n=1}^{\infty} u_n z^n$$

which is regular in $|z| < 1$ and for which $\mathrm{Re}\{F(z)\} > 0$, then

$$A(\theta) = \lim_{r \to 1} \{(\tfrac{1-r}{1+r})(F(re^{i\theta}) - e^{i\alpha} + 1)\} = \lim_{r \to 1} \{(\tfrac{1-r}{1+r}) F(re^{i\theta})\}$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^2 d\theta = \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta}) - e^{i\alpha} + 1|^2\, d\theta = 1 + \sum_{n=1}^{\infty} |u_n|^2 r^{2n}$$

and so by lemma 4, we have

$$(4.1.17) \qquad \lim_{r \to 1} (1-r) \sum_{n=1}^{\infty} |u_n|^2 r^n = 4\Sigma \alpha_\nu^{\ 2}$$

where α_ν's have the same meaning as in lemma 4.

4.2. In this section we shall prove the following :

Theorem 1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(\alpha)$, then

(4.2.1) $\quad (n-1)^2|a_n|^2 \leq 4\cos^2\alpha\,[1 + \sum_{k=2}^{n-1} k|a_k|^2], \quad n = 2, 3, 4, \ldots .$

Proof : If

$$F(z) = \frac{e^{i\alpha} z f'(z)}{f(z)} = \frac{1}{2}u_o + \sum_{n=1}^{\infty} u_n z^n$$

where $u_o = 2e^{i\alpha}$ and $|z| < 1$, then it can be easily seen that the coefficients of F are linked to those of $f(z)$ by the following system of equations

(4.2.2) $$e^{i\alpha} a_n (n+1) = \sum_{k=1}^{n} a_k u_{n-k} \; ; \; a_1 = 1$$

Now, for $0 \leq r < 1$ and $n = 1,2,3,\ldots,$ we have ([64],p.86)

(4.2.3) $$\pi u_n r^n = \int_o^{2\pi} e^{-in\theta} \operatorname{Re}\{F(re^{i\theta})\} d\theta$$

Since $\operatorname{Re}\{F(z)\} > 0$, by Herglotz's theorem ([1], p.40) there exists a non-decreasing positive function $\mu_f(t)$ defined on the closed interval $[0,2\pi]$ such that

(4.2.4) $$F(z) = \int_o^{2\pi} \frac{z + e^{it}}{e^{it} - z}\, d\mu_f(t) + i \sin\alpha$$

Since $F(0) = e^{i\alpha}$, (4.2.4) implies that

(4.2.5) $$\cos\alpha = \int_o^{2\pi} d\mu_f(t)$$

Hence by (4.2.3) and (4.2.4) we obtain for $n \geq 1$

$$\pi u_n r^n = \int_0^{2\pi} e^{-in\theta}\{ \int_0^{2\pi} \frac{(1-r^2)d\mu_f(t)}{|1-ze^{-it}|^2} \} d\theta$$

$$= 2\pi r^n \int_0^{2\pi} e^{-int}\, d\mu_f(t)$$

Therefore, for $n \geq 1$

(4.2.6) $$u_n = 2 \int_0^{2\pi} e^{-int}\, d\mu_f(t)$$

From (4.2.2) and (4.2.6) we have

(4.2.7) $$e^{i\alpha}\, a_n (n+1) = 2 \sum_{k=1}^{n} a_k \int_0^{2\pi} e^{-i(n-k)t}\, d\mu_f(t)$$

$$= 2 \int_0^{2\pi} e^{-int} (\sum_{k=1}^{n} a_k e^{ikt}) d\mu_f(t)$$

$$= 2 \int_0^{2\pi} e^{-int}\, s_n(t)\, d\mu_f(t)$$

Applying Schwarz's inequality in (4.2.7), we obtain

(4.2.8) $$|a_n|^2\, (n+1)^2 \leq 4\{\int_0^{2\pi} d\mu_f(t)\}\{\int_0^{2\pi} |s_n(t)|^2\, d\mu_f(t)\}$$

From lemma 2 and (4.2.8) we finally have, with $a_1 = 1$

$$(n+1)^2 |a_n|^2 \leq 4 \sum_{k=1}^{n} k|a_k|^2 \cos^2\alpha$$

$$= 4 \sum_{k=1}^{n-1} k|a_k|^2 \cos^2\alpha + 4n|a_n|^2 \cos^2\alpha$$

from which (4.2.1) follows.

From theorem 1, we get an important result which provides a sharper estimate for the coefficients than the estimate $|a_n| \leq n$ obtained by Špaček [62] for the class of univalent α-spiral functions. We state the result as a theorem.

Theorem 1'. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to the class $S(\alpha)$ then

$$|a_n| \leq n \cos\alpha \quad \text{for } n = 2, 3, 4, \ldots . \tag{4.2.9}$$

Proof : That (4.2.9) holds for $n = 2$ is evident from theorem 1. Suppose (4.2.9) holds for $n = 2,3,\ldots,m$. From theorem 1, we find that

$$m^2|a_{m+1}|^2 \leq 4[1+2|a_2|^2 + 3|a_3|^2 + \ldots + m|a_m|^2] \cos^2\alpha$$

$$\leq 4[1^3 + 2^3 + 3^3 + \ldots + m^3] \cos^2\alpha$$

$$= 4\left(\frac{m(m+1)}{2}\right)^2 \cos^2\alpha$$

Hence, $|a_{m+1}| \leq (m+1)\cos\alpha$.

Thus (4.2.9) holds for $n = m+1$ also. Hence by induction it holds for all $n = 2,3,4,\ldots$.

An important consequence of (4.2.9) is that, if $|\alpha| = \frac{\pi}{2}$ then $|a_n| = 0$, i.e., $a_n = 0$ for $n = 2,3,4,\ldots$. This implies that in this case the only univalent α-spiral function is the identity function $f(z) = z$. With a slight modification in (4.2.9) one may prove that if $|\alpha| = \frac{\pi}{2}$, then in general, the univalent α-spiral function $f(z)$ for which $f(0) = 0$, has the form $f(z) = a_1 z$, where a_1 is a constant.

Theorem 2 : If $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n \in S_\rho(\alpha)$, then

$$(4.2.10) \quad \sum_{n=mk+1}^{(m+1)k} (n-1)^2 |a_n|^2 \le \left[\frac{k}{(m-1)!} \prod_{\mu=0}^{m-1} \left(\mu + \frac{2(1-\rho)\cos^2\alpha}{k}\right)\left\{1+\left(\frac{2(1-\rho)\cos\alpha\sin\alpha}{\mu k+2(1-\rho)\cos^2\alpha}\right)^2\right\}^{\frac{1}{2}}\right]^2$$

and

$$(4.2.11) \quad \sum_{n=mk+1}^{(m+1)k} (n-\rho)|a_n|^2 \le (mk+1-\rho)\left[\frac{1}{m!} \prod_{\mu=0}^{m-1} \left(\mu + \frac{2(1-\rho)\cos^2\alpha}{k}\right)\left\{1+\left(\frac{2(1-\rho)\cos\alpha\sin\alpha}{\mu k+2(1-\rho)\cos^2\alpha}\right)^2\right\}^{\frac{1}{2}}\right]^2$$

Proof : Since $f(z) \in S_\rho(\alpha)$, therefore there exists a bounded regular function $\omega(z)$ such that $\omega(0) = 0$ and $|\omega(z)| \le 1$ and

$$(4.2.12) \quad \frac{e^{i\alpha}\sec\alpha \, zf'(z)}{f(z)} - i\tan\alpha = \frac{1+(1-2\rho)\omega(z)}{1-\omega(z)}$$

By direct computation we have

$$(4.2.13) \quad \omega(z) = b_k z^k + b_{k+1} z^{k+1} + \ldots, \text{ where } b_k = \frac{k a_{k+1} e^{i\alpha}}{2(1-\rho)\cos\alpha}, \text{ etc.}$$

Write (4.2.12) in the form

$$(4.2.14) \left(\sum_{n=k}^{\infty} b_n z^n\right)\left(2(1-\rho)z\cos\alpha + \sum_{n=k+1}^{\infty} (ne^{i\alpha}+e^{-i\alpha} - 2\rho\cos\alpha)a_n z^n\right) = e^{i\alpha}\sum_{n=K+1}^{\infty}(n-1)a_n z^n$$

Equating the coefficients of $z^{k+1}, z^{k+2}, \ldots, z^{2k}$ in (4.2.14) we obtain

$$(4.2.15) \quad 2(1-\rho)b_n \cos\alpha = n\, a_{n+1} e^{i\alpha}$$

for $n = k+1, k+2, k+3, \ldots, 2k$.

Since $|\omega(z)| \le 1$, it follows that $\sum_{n=k}^{\infty} |b_n|^2 \le 1$, and therefore

$$(4.2.16) \qquad \sum_{n=k}^{2k-1} |b_n|^2 \leq 1$$

Hence, from (4.2.15) and (4.2.16) we find that

$$(4.2.17) \qquad \sum_{n=k+1}^{2k} (n-1)^2 |a_n|^2 \leq 4(1-\rho)^2 \cos^2\alpha$$

We rewrite (4.2.14) as follows

$$(4.2.18) \sum_{n=k+1}^{p} e^{i\alpha}(n-1)a_n z^n + \sum_{n=p+1}^{\infty} d_n z^n = \omega(z)[2(1-\rho)z \cos\alpha + \sum_{n=k+1}^{p-k} \{ne^{i\alpha} + e^{-i\alpha} - 2\rho\cos\alpha\} a_n z^n]$$

The constants d_n occuring in (4.2.18) are determined by the identity (4.2.12). Now following the method of Clunie [8], we obtain that

$$\sum_{n=k+1}^{p} (n-1)^2 |a_n|^2 r^{2n} + \sum_{n=p+1}^{\infty} |d_n|^2 r^{2n} \leq 4r^2(1-\rho)^2\cos^2\alpha + \sum_{n=k+1}^{p-k} |ne^{i\alpha} + e^{-i\alpha} - 2\rho\cos\alpha|^2 |a_n|^2 r^{2n}$$

Making $r \to 1$, finally, we have

$$(4.2.19) \sum_{n=k+1}^{p} (n-1)^2 |a_n|^2 \leq 4(1-\rho)^2\cos^2\alpha + \sum_{n=k+1}^{p-k} |ne^{i\alpha} + e^{-i\alpha} - 2\rho\cos\alpha|^2 |a_n|^2$$

Simplifying the quantity under the modulus sign on the right hand side of (4.2.19), we obtain

$$(4.2.20) \sum_{n=k+1}^{p} (n-1)^2 |a_n|^2 \leq 4(1-\rho)^2\cos^2\alpha + \sum_{n=k+1}^{p-k} \{(n-1)^2 + 4(1-\rho)(n-\rho)\cos^2\alpha\} |a_n|^2$$

$$= 4(1-\rho)\cos^2\alpha[(1-\rho) + \sum_{n=k+1}^{p-k} (n-\rho)|a_n|^2] + \sum_{n=k+1}^{p-k} (n-1)^2 |a_n|^2$$

Now by induction, we shall establish the inequalities (4.2.10) and (4.2.11).

For m = 1, (4.2.10) can be written as following.

$$\sum_{n=k+1}^{2k} (n-1)^2|a_n|^2 \leq [k\, \frac{2(1-\rho)\cos^2\alpha}{k}\{1 + (\frac{2(1-\rho)\cos\alpha\sin\alpha}{2(1-\rho)\cos^2\alpha})^2\}^{\frac{1}{2}}]^2$$

$$= 4(1-\rho)^2\cos^2\alpha$$

which evidently follows from (4.2.17). Thus the inequality (4.2.10) is true for m = 1. Assuming it is true for m = 1,2,3,..., q-1, we have from (4.2.20) that

$$\sum_{n=qk+1}^{(q+1)k} (n-1)^2|a_n|^2 \leq 4(1-\rho)\cos^2\alpha\ [(1-\rho) + \sum_{n=k+1}^{qk} (n-\rho)|a_n|^2]$$

$$= 4(1-\rho)\cos^2\alpha[(1-\rho) + \sum_{m=1}^{q-1} (\sum_{n=mk+1}^{(m+1)k} (n-\rho)|a_n|^2)]$$

$$\leq 4(1-\rho)\cos^2\alpha[(1-\rho) + \sum_{m=1}^{q-1} \frac{mk+1-\rho}{m^2k^2}\{\sum_{n=mk+1}^{(m+1)k} (n-1)^2|a_n|^2\}]$$

$$\leq 4(1-\rho)\cos^2\alpha[(1-\rho) + \sum_{m=1}^{q-1} (mk+1-\rho)\{\frac{1}{m!}\prod_{\mu=0}^{m-1} (\mu + \frac{2(1-\rho)\cos^2\alpha}{k})(1 + (\frac{2(1-\rho)\cos\alpha\sin\alpha}{\mu k+2(1-\rho)\cos^2\alpha})^2)^{\frac{1}{2}}\}^2]$$

The last inequality is obtained from the hypothesis on (4.2.10) from m =1,2,3,...,q-1. Finally, by using lemma 3, we obtain

$$\sum_{n=qk+1}^{(q+1)k} (n-1)^2|a_n|^2 \leq [\frac{k}{(q-1)!}\prod_{\mu=0}^{q-1}\{\mu + \frac{2(1-\rho)\cos^2\alpha}{k}\}\{1 + (\frac{2(1-\rho)\cos\alpha\sin\alpha}{\mu k+2(1-\rho)\cos^2\alpha})^2\}^{\frac{1}{2}}]^2$$

This establishes (4.2.10).

For proving (4.2.11), we note that for $m = 1$, the inequality (4.2.11) takes the form

$$\text{(4.2.21)} \quad \sum_{n=K+1}^{2k} (n-\rho)|a_n|^2 \leq (k+1-\rho)\left[\frac{2(1-\rho)\cos^2\alpha}{k} \frac{1}{\cos\alpha}\right]^2 = \frac{k+1-\rho}{k^2} 4(1-\rho)^2 \cos^2\alpha$$

Now we shall show that (4.2.21) actually holds. From (4.2.17) we have

$$\text{(4.2.22)} \quad \sum_{n=k+1}^{2k} (n-\rho)|a_n|^2 = \frac{(k+1-\rho)}{k^2} \sum_{n=k+1}^{2k} \frac{k^2(n-\rho)}{(k+1-\rho)}|a_n|^2$$

$$\leq \frac{(k+1-\rho)}{k^2} \sum_{n=k+1}^{2k} (n-1)^2|a_n|^2$$

$$\leq \frac{(k+1-\rho)}{k^2} 4(1-\rho)^2 \cos^2\alpha$$

Thus we get (4.2.21). Next, assume that (4.2.11) is true for $m=1,2,3,\ldots,q-1$, then we obtain

$$\text{(4.2.23)} \quad \sum_{n=qk+1}^{(q+1)k} (n-\rho)|a_n|^2 = \frac{(qk+1-\rho)}{q^2k^2} \sum_{n=qk+1}^{(q+1)k} \frac{(n-\rho)q^2k^2}{qk+1-\rho} |a_n|^2$$

$$\leq \frac{qk+1-\rho}{q^2k^2} \sum_{n=qk+1}^{(q+1)k} (n-1)^2|a_n|^2$$

From (4.2.10) and (4.2.23), (4.2.11) follows for $m = q$. Thus by induction the truth of (4.2.11) is established, since (4.2.11) holds for $m = 1$. This completes the proof of theorem 2.

As direct consequences of theorem 2, we get the following main results, which we state as theorems in view of their importance.

Theorem 3. *If* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n \in S_\rho(\alpha)$, *then*

$$(4.2.24) \quad |a_n| \leq \frac{k}{(n-1)(m-1)!} \prod_{\mu=0}^{m-1} \left(\mu + \frac{2(1-\rho)\cos^2\alpha}{k}\right)\left\{1+\left(\frac{2(1-\rho)\cos\alpha\sin\alpha}{\mu k+2(1-\rho)\cos^2\alpha}\right)^2\right\}^{\frac{1}{2}}$$

where $mk+1 \leq n \leq (m+1)k$, $m = 1, 2, 3, \ldots$.

Theorem 4. *If* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ *is convex, univalently regular function of order* ρ *for* $|z| < 1$, *then for* $mk+1 \leq n \leq (m+1)k, m=1,2,3,\ldots,$ *we have*

$$(4.2.25) \qquad |a_n| \leq \frac{k}{n(n-1)(m-1)!} \prod_{\mu=0}^{m-1} \left(\mu + \frac{2(1-\rho)}{k}\right)$$

Theorem 4 follows immediately from theorem 3 by using the wellknown fact that $f(z)$ is convex of order ρ, if and only if, $zf'(z)$ is starlike of order ρ.

4.3. In this section we shall give some applications of theorems 4 and 5.

(a) Taking $\alpha = 0$ and $\rho = 0$, so that $f(z)$ is starlike, we find that following theorem of MacGregor [36] follows as a corollary to theorem 3.

Theorem [MacGregor]. *If* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ *is regular, univalent and starlike for* $|z| < 1$, *then*

$$|a_n| \leq \frac{k}{(n-1)(m-1)!} \prod_{\mu=0}^{m-1} \left(\mu + \frac{2}{k}\right)$$

where $mk+1 \leq n \leq (m+1)k$, $m = 1,2,3,\ldots$.

(b) Taking $\rho = 0$ in theorem 4, the following result of MacGregor [36] follows :

Theorem [MacGregor] : *If* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ *is univalent, regular and convex for* $|z| < 1$, *then*

$$|a_n| \leq \frac{k}{n(n-1)(m-1)!} \prod_{\mu=0}^{m-1} (\mu + \frac{2}{k})$$

where $mk+1 \leq n \leq (m+1)k$, $m = 1,2,3,\ldots$.

(c) If we take $\rho = 0$ and $k = 1$ in theorem 3, we get the following theorem of Zamorski [69].

Theorem [Zamorski] : If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_o(\alpha)$, then

$$|a_n| \leq \prod_{\mu=0}^{n-2} \frac{|2e^{-i\alpha}\cos\alpha + \mu|}{\mu + 1}$$

(d) If we take $\alpha = 0$ and $k = 1$ we get the following theorem of Robertson [52] which has also been obtained by Schild [59] recently.

Theorem [[Robertson], [Schild]] : If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_\rho(0)$ then

$$|a_n| \leq \frac{1}{(n-1)!} \prod_{\mu=0}^{n} (\mu - 2\rho)$$

(e) More generally, if we take $k = 1$ in theorem 3, the following theorem of Libera [32] follows.

Theorem [Libera] : If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_\rho(\alpha)$, then

$$|a_n| \leq \prod_{\mu=0}^{n-2} \frac{|2(1-\rho)\cos\alpha\ e^{-i\alpha} + \mu|}{\mu + 1} \quad ; n = 2,3,4,\ldots .$$

For $k = 2$, theorem 3, yields that

(4.3.1) $$|a_{2n+1}| \leq \frac{1}{n!} \prod_{\mu=0}^{n-1} \{\mu^2 + 2(1-\rho)\mu \cos^2\alpha + (1-\rho)^2 \cos^2\alpha\}^{\frac{1}{2}}$$ and

(4.3.2) $$|a_{2n+2}| \leq \frac{2}{(2n+1)(n-1)!} \prod_{\mu=0}^{n-1} \{\mu^2 + 2(1-\rho)\cos^2\alpha + (1-\rho)^2 \cos^2\alpha\}^{\frac{1}{2}}$$

If we take $k = 2, \rho = 0$ and $\alpha = 0$ in (4.3.1) then the following estimate due to Golusin [15] follows.

$$|a_{2n+1}| \leq 1$$

Similarly, if we take $\rho = 0$, $\alpha = 0$ and $k = 2$ in (4.3.2), we get the following result of MacGregor [36]

$$|a_{2n+2}| \leq \frac{2n}{2n+1}$$

4.4 In this section we shall prove a growth theorem for $f(z) \in S(\alpha)$.

Theorem 5: If $f(z) \in S(\alpha)$ then

$$\int_0^{2\pi} \log \left|\frac{f(re^{i\theta})}{r}\right| d\mu_f(\theta) = 2 \cos \alpha \Sigma \alpha_\nu^2 \log \left(\frac{1}{1-r}\right)$$

as $r \to 1$, where α_ν's are defined as in lemma 4.

Proof : Using (4.2.1) and (4.2.6), we obtain for every r, $0 < r < 1$,

$$2\int_0^{2\pi} F(re^{i\theta}) d\mu_f(\theta) = 2 \int_0^{2\pi} \frac{u_0}{2} d\mu_f(\theta) + \sum_{n=1}^{\infty} u_n \int_0^{2\pi} r^n e^{in\theta} d\mu_f(\theta)$$

$$= 2 e^{i\alpha} \int_0^{2\pi} d\mu_f(\theta) + \sum_{n=1}^{\infty} u_n r^n \int_0^{2\pi} e^{in\theta} d\mu_f(\theta)$$

$$= 2e^{i\alpha} \cos\alpha + \sum_{n=1}^{\infty} |u_n|^2 r^n$$

Hence

(4.4.1) $$2 \int_0^{2\pi} \mathrm{Re}\{F(re^{i\theta}) - \cos \alpha\} d\mu_f(\theta) = \sum_{n=1}^{\infty} |u_n|^2 r^n$$

and

$$(4.4.2) \qquad 2 \int_0^{2\pi} \mathrm{Im}\{F(re^{i\theta}) - \sin\alpha\} d\mu_f(\theta) = 0$$

Also, if $G = zf'(z)/f(z)$ then

$$\mathrm{Re}\{F(re^{i\theta})\} = \cos\alpha \ \mathrm{Re}\{G\} - \sin\alpha \ \mathrm{Im}\{G\}$$

$$\mathrm{Im}\{F(re^{i\theta})\} = \cos\alpha \ \mathrm{Im}\{G\} + \sin\alpha \ \mathrm{Re}\{G\}$$

The above simultaneous equations yield,

$$(4.4.3) \quad \mathrm{Re}\{G\} = 1 + \cos\alpha[\mathrm{Re}\{F(re^{i\theta})\} - \cos\alpha] + \sin\alpha[\mathrm{Im}\{F(re^{i\theta})\} - \sin\alpha]$$

Equations (4.4.1), (4.4.2) and (4.4.3) together yield

$$2\int_0^{2\pi} \mathrm{Re}\{G\} d\mu_f(\theta) = 2\cos\alpha \int_0^{2\pi} [\mathrm{Re}\{F(re^{i\theta})\} - \cos\alpha] d\mu_f(\theta) + 2\int_0^{2\pi} d\mu_f(\theta)$$

$$= 2 \cos\alpha \sum_{n=1}^{\infty} |u_n|^2 r^n + 2\int_0^{2\pi} d\mu_f(\theta)$$

From this equation, we have

$$2\int_0^{2\pi} \mathrm{Re}\left\{e^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})}\right\} d\mu_f(\theta) = \cos\alpha \sum_{n=1}^{\infty} |u_n|^2 r^{n-1} + \frac{2}{r}\int_0^{2\pi} d\mu_f(\theta)$$

Integrating the above equation with respect to r from 0 to r, we obtain

$$(4.4.4) \qquad 2\int_0^{2\pi} \log\left|\frac{f(re^{i\theta})}{r}\right| d\mu_f(\theta) = \cos\alpha \sum_{n=1}^{\infty} \frac{|u_n|^2 r^n}{n}$$

In view of lemma 4, the equation (4.4.4) yields the required result of the theorem.

CHAPTER 5

Radius Of Convexity And The Coefficient Problem Of Meromorphically Generalized Close-TO-Convex Functions

5.1. In this chapter we generalize a class of meromorphically close-to-convex functions introduced by Libera [31] and study the radius of convexity problem and the coefficient problem which will include the results due to Libera. We begin by reproducing some definitions given by Libera.

Definition 1. Let

$$(5.1.1) \qquad F(z) = \frac{e^{i\alpha}}{z} + b_0 + b_1 z + b_2 z^2 + \ldots , \ (\alpha \text{ real})$$

be meromorphic function in the unit disc D with simple pole at $z = 0$ with the residue $e^{i\alpha}$. The function $F(z)$ is said to be starlike of order σ, $0 \leq \sigma \leq 1$, if and only if,

$$(5.1.2) \qquad \mathrm{Re}\ \{\frac{-zF'(z)}{F(z)}\} \geq \sigma \quad , z \in D$$

This class of functions is denoted by Σ^*_σ .

Definition 2. Denote by $B(\lambda,\sigma)$, $0 \leq \lambda, \sigma \leq 1$, the family of functions

$$(5.1.3) \qquad f(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

which are regular in D except for a simple pole at $z = 0$ and together with some $F \in \Sigma^*_\sigma$, such that

$$(5.1.4) \qquad \mathrm{Re}\ \{\frac{zf'(z)}{F(z)}\} \geq \lambda \ , z \in D$$

Then $f(z)$ is said to be meromorphically close-to-convex of order λ and

type σ with respect to the function $F(z)$.

The class $B(\lambda,\sigma)$ can further be extended as follows. Since

$$(5.1.5) \qquad \left.\frac{zf'(z)}{F(z)}\right|_{z=0} = -e^{i\alpha}, \quad -\cos\alpha \geq \lambda \geq 0$$

We have

$$(5.1.6) \qquad \operatorname{Re}\left\{-\sec\alpha\frac{zf'(z)}{F(z)} + i\tan\alpha\right\} \geq -\lambda\sec\alpha$$

The left hand side of (5.1.6) has value 1 at $z = 0$. Now we define the class $B(\beta,\lambda,\sigma)$, $0 \leq \lambda$, $\sigma \leq 1$, and $\beta \geq 1$, of all functions $F(z)$ and $f(z)$ having representations (5.1.1) and (5.1.3) respectively, such that

$$(5.1.7) \qquad \left|\left\{\frac{-zf'(z)\sec\alpha}{F(z)} + i\tan\alpha + \lambda\sec\alpha\right\}(1+\lambda\sec\alpha)^{-1} - \beta\right| < \beta$$

for all $|z| < 1$.

The family of functions $f \in B(\lambda,\sigma)$ has been the subject of recent investigations by R.J. Libera [31] and the family of functions $f \in B(0,0)$ has been studied by Libera and Robertson [28]. We state here a lemma due to Goel [14], which we shall need.

Lemma [Goel]. Let $P(z) = 1 + p_1 z + p_2 z^2 + \dots + p_n z^n + \dots$, be regular in D and satisfy the condition

$$|P(z) - \beta| < \beta \quad (\beta \geq 1,\ |z| < 1)$$

then

$$P(z) = \frac{1+\omega(z)}{1 + \left(\frac{1-\beta}{\beta}\right)\omega(z)}$$

for some $\omega(z)$ satisfying the conditions of Schwarz's lemma (see [Lemmas 1 and 2], chapter 2), and further

$$\frac{1-|z|}{1+(1-\frac{1}{\beta})|z|} \leq |P(z)| \leq \frac{1+|z|}{1-(1-\frac{1}{\beta})|z|}$$

and

$$\left|\frac{zP'(z)}{P(z)}\right| \leq \frac{|z|}{1-\frac{1}{\beta}|z| - (1-\frac{1}{\beta})|z|^2}$$

5.2. The Radius Of Convexity. In this section we shall derive the largest region $\{|z| < r\}$ of convexity for $f \in B(\beta,\lambda,\sigma)$ and prove the following :

Theorem 1. *If $f(z) \in B(\beta,\lambda,\sigma)$ then f is convex and consequently univalent in disc $|z| < r$, $r > r(\beta,\lambda,\sigma)$, $r(\beta,\lambda,\sigma)$ being the smallest positive root of the polynomial equation.*

(5.2.1) $$r^3[(1-\tfrac{1}{\beta})(1+\lambda\ \sec\alpha)+\lambda\ \sec\alpha](1-2\sigma)-r^2[2(1-\sigma)\{(1-\tfrac{1}{\beta})(1+\lambda\ \sec\alpha) +\lambda\ \sec\alpha\}+ (2-\tfrac{1}{\beta})(1+\lambda\ \sec\alpha) + (2\sigma-1)] + r(2\sigma-3) + 1 = 0$$

Proof. Let

(5.2.2) $$P(z) = \frac{-\sec\alpha\ \frac{zf'(z)}{F(z)} + i\ \tan\alpha+\lambda\ \ \sec\alpha}{1 + \lambda\ \sec\alpha}$$

then $P(z)$ satisfies the conditions of lemma . Differentiating (5.2.2) and simplifying we get

$$(5.2.3)\quad -\mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\}=-\mathrm{Re}\{\frac{zF'(z)}{F(z)}\}-\max\left|\frac{zP'(z)}{P(z)}\cdot\frac{P(z)}{P(z)-\frac{\lambda\sec\alpha+i\tan\alpha}{1+\lambda\sec\alpha}}\right|$$

But from lemma as used in(see([14],p.p.104-116)) we have

$$(5.2.4)\quad \left|\frac{\frac{zP'(z)}{P(z)}}{1-(\frac{\lambda\sec\alpha+i\tan\alpha}{1+\lambda\sec\alpha})\frac{1}{P(z)}}\right| \le \frac{(2-\frac{1}{\beta})|z|}{1-\frac{|z|}{\beta}-(1-\frac{1}{\beta})|z|^2}\cdot\frac{1+\lambda\sec\alpha}{1+\lambda\sec\alpha-\frac{\lambda(1-|z|)\sec\alpha}{1+(1-\frac{1}{\beta})|z|}}$$

Further, since $F(z)$ is starlike meromorphic functinn of order σ, we have [39]

$$(5.2.5)\qquad \mathrm{Re}\,\{\frac{-zF'(z)}{F(z)}\}\ge\frac{1-(1-2\sigma)r}{1+r}\quad;\ 0\le|z|<r$$

Hence from (5.2.4) and (5.2.5) we have for $|z|=r$

$$(5.2.6)\quad -\mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\}\ge\frac{(1-(1-2\sigma)r)}{1+r}-\frac{(2-\frac{1}{\beta})r(1+\lambda\sec\alpha)}{(1-r)[1+\{(1-\frac{1}{\beta})(1+\lambda\sec\alpha)+\lambda\sec\alpha\}r]}$$

$$=\frac{r^3[(1-\frac{1}{\beta})(1+\lambda\sec\alpha)+\lambda\sec\alpha](1-2\sigma)-r^2[2(1-\sigma)\{(1-\frac{1}{\beta})(1+\lambda\sec\alpha)+\lambda\sec\alpha\}+(2-\frac{1}{\beta})(1+\lambda\sec\alpha)+(2\sigma-1)]+r[2\sigma-3]+1}{(1-r^2)[1+\{(1-\frac{1}{\beta})(1+\lambda\sec\alpha)+\lambda\sec\alpha\}r]}$$

We note that the denominator of the expression is positive for $\beta\ge1$ and $0\le r<1$. Hence the smallest radius of convexity region for $f(z)\in B(\beta,\lambda,\sigma)$ can be obtained whenever the numerator is positive. Denote the numerator of (5.2.6) by $U(r)$, then $U(0)=1$ and $U(1)=-2(2-\frac{1}{\beta})(1+\lambda\sec\alpha)\le0$. Therefore the smallest positive root lies between 0 and 1,hence the result of the theorem follows.

From theorem 1 we obtain the following result of Libera [31] by making $\beta\to\infty$ in theorem 1.

Theorem [Libera] : If $f \in B(\lambda,\sigma)$ with respect to F, then f is convex and consequently univalent in a disc $|z| < r$, $r \geq r(\lambda,\sigma)$, $r(\lambda,\sigma)$ being the smallest positive root of the polynomial equation

$$r^3[1+2\lambda\sec\alpha](1-2\sigma)-r^2[3+6\lambda\sec\alpha-4\sigma\lambda\sec\alpha] + (2\sigma-3)r + 1 = 0$$

5.3 Coefficient Bounds. The following theorem gives an upper estimate for the coefficients of f in $B(\beta,\lambda,\sigma)$.

Theorem 2. If $f(z) \in B(\beta,\lambda,\sigma)$ with respect to $F(z)$, having the Laurent expansion (5.1.3) and (5.1.1) respectively, and further if

$$\sum_{k=1}^{\infty} k|a_k|^2 \leq 1 \tag{5.3.1}$$

then

$$\begin{aligned}(n|a_n|+|b_n|)^2 \leq {} & [(2-\tfrac{1}{\beta})^2(\lambda+\cos\alpha)^2+\{|(1-\tfrac{1}{\beta})(\tfrac{\lambda-i\sin\alpha}{\beta} - \tfrac{2\lambda+e^{-i\alpha}}{\beta} - \\ & 2(\lambda+\cos\alpha)|\} \sum_{k=1}^{n-1} 2k|a_k||b_k|+\sum_{k=0}^{n-1}|b_k|^2\{(\tfrac{\lambda^2+\sin^2\alpha}{\beta^2})-\tfrac{2(2\lambda^2+\lambda\cos\alpha+\sin^2\alpha)}{\beta}+ \\ & 4\lambda(\lambda+\cos\alpha)\}+4n|a_n||b_n|-\tfrac{1}{\beta}(2-\tfrac{1}{\beta})\sum_{k=0}^{n-1}|a_k|^2 k^2\end{aligned} \tag{5.3.2}$$

Proof. The proof of this theorem depends on the method of Clunie as used by Libera and Robertson [28]. Let

$$P(z) = [\frac{-\sec\alpha\ zf'(z)}{F(z)} + i\tan\alpha+\lambda\ \sec\alpha\ ][1 + \lambda\sec\alpha]^{-1}$$

By lemma , we have

$$\frac{-\sec\alpha z f'(z)}{F(z)} = \frac{e^{-i\alpha}\sec\alpha + [(2\lambda\sec\alpha + e^{i\alpha}\sec\alpha) - (\lambda + i\sin\alpha)\frac{\sec\alpha}{\beta}]\omega(z)}{1-(1-\frac{1}{\beta})\omega(z)}$$

Hence by direct computation, we have

$$\omega(z)[\{-1+\frac{1}{\beta}+(\frac{\lambda}{\beta}+\frac{i\sin\alpha}{\beta}-2\lambda-e^{i\alpha})e^{i\alpha}\}+\text{terms containing } z]=b_0e^{-i\alpha}z+(b_1e^{-i\alpha}+a_1)z^2+\ldots$$

and so

$$\omega(0) = 0 \text{ and } \omega'(0) = \frac{-b_0\, e^{-2i\alpha}}{-\frac{1}{\beta}(\lambda+i\sin\alpha)+(2\lambda+e^{i\alpha})+(1-\frac{1}{\beta})e^{-i\alpha}}$$

We further observe that, when $a_{-1} = 1$ and $b_{-1} = e^{i\alpha}$ then

$$\omega(z)[\sum_{k=-1}^{\infty}\{ka_k(1-\frac{1}{\beta})+(\frac{\lambda+i\sin\alpha}{\beta}-2\lambda-e^{i\alpha})b_k\}z^{k+1}=\sum_{k=0}^{\infty}(ka_k+b_ke^{-i\alpha})z^{k+1}$$

Thus we may write

$$\omega(z)\sum_{k=-1}^{n-1}[ka_k(1-\frac{1}{\beta})+\{\frac{1}{\beta}(\lambda+i\sin\alpha)-(2\lambda+e^{i\alpha})\}b_k]z^{k+1}=\sum_{k=0}^{n}(b_ke^{-i\alpha}+ka_k)z^{k+1}+\sum_{k=n+2}^{\infty}c_kz^k$$

The above equation implies that

$$\sum_{k=-1}^{n-1}|ka_k(1-\frac{1}{\beta})+\{\frac{1}{\beta}(\lambda+i\sin\alpha)-(2\lambda+e^{i\alpha})\}b_k|^2 r^{2(k+1)}$$

$$\geq \frac{1}{2\pi}\int_0^{2\pi}|\sum_{k=0}^{n}(b_k e^{-i\alpha}+k a_k)(re^{i\theta})^{k+1}|^2 d\theta$$

$$= \sum_{k=0}^{n}|k a_k + b_k e^{-i\alpha}|^2 r^{2k+2}$$

Making $r \to 1$, we get

$$\sum_{k=0}^{n} |ka_k + b_k e^{-i\alpha}|^2 \le \sum_{k=-1}^{n-1} |ka_k(1-\tfrac{1}{\beta}) + \{\tfrac{1}{\beta}(\lambda+i\sin\alpha)-(2\lambda+e^{i\alpha})\}b_k|^2$$

(5.3.3) $$|na_n+b_n e^{-i\alpha}|^2 \le |-(1-\tfrac{1}{\beta}) + \{\tfrac{\lambda+i\sin\alpha}{\beta} - (2\lambda+e^{i\alpha})\}e^{i\alpha}|^2 + \sum_{k=0}^{n-1} [|ka_k(1-\tfrac{1}{\beta}) + \{\tfrac{1}{\beta}(\lambda+i\sin\alpha)-(2\lambda+e^{i\alpha})\}b_k|^2 - |ka_k + b_k e^{-i\alpha}|^2]$$

Now we note that

(5.3.4) $$|na_n+b_n e^{-i\alpha}|^2 = n^2|a_n|^2 + |b_n|^2 + 2\mathrm{Re}\{n\, e^{i\alpha}\, a_n \bar{b}_n\}$$

(5.3.5) $$|-(1-\tfrac{1}{\beta})e^{-i\alpha} + \tfrac{\lambda+i\sin\alpha}{\beta} - 2\lambda - e^{i\alpha}|^2 = (2-\tfrac{1}{\beta})^2(\lambda+\cos\alpha)^2$$

(5.3.6) $$|k\, a_k(1-\tfrac{1}{\beta}) + \{\tfrac{\lambda+i\sin\alpha}{\beta} - (2\lambda+e^{i\alpha})\}b_k|^2 = k^2|a_k|^2(1-\tfrac{1}{\beta})^2 + |b_k|^2\{\tfrac{\lambda^2+\sin^2\alpha}{\beta^2} - 2[\tfrac{2\lambda^2+\lambda\cos\alpha+\sin^2\alpha}{\beta}] + 4\lambda^2+4\lambda\cos\alpha+1\} + 2\mathrm{Re}\{ka_k\bar{b}_k(1-\tfrac{1}{\beta})[\tfrac{\lambda-i\sin\alpha}{\beta} - (2\lambda+e^{-i\alpha})]\}\}$$

and

(5.3.7) $$|ka_k+b_k e^{-i\alpha}|^2 = k^2|a_k|^2 + |b_k|^2 + 2\,\mathrm{Re}\{k\, a_k\, \bar{b}_k\, e^{i\alpha}\}$$

Hence from (5.3.6) and (5.3.7) we have

(5.3.8) $$\{|ka_k(1-\tfrac{1}{\beta}) + \{\tfrac{\lambda+i\sin\alpha}{\beta} - (2\lambda+e^{i\alpha})\}b_k|^2 - |ka_k+b_k e^{i\alpha}|^2\} = -k^2|a_k|^2\,\tfrac{1}{\beta}(2-\tfrac{1}{\beta}) + |b_k|^2(\tfrac{\lambda^2+\sin^2\alpha}{\beta^2} - 2(\tfrac{2\lambda^2+\lambda\cos\alpha+\sin^2\alpha}{\beta}) + 4\lambda(\lambda+\cos\alpha)\} + 2\mathrm{Re}\{ka_k\bar{b}_k[(\tfrac{\lambda-i\sin\alpha}{\beta})(1-\tfrac{1}{\beta}) + \tfrac{2\lambda+e^{-i\alpha}}{\beta} - 2(\lambda+\cos\alpha)]\}.$$

Making $r \to 1$, we get

$$\sum_{k=0}^{n} |ka_k + b_k e^{-i\alpha}|^2 \le \sum_{k=-1}^{n-1} |ka_k(1-\tfrac{1}{\beta}) + \{\tfrac{1}{\beta}(\lambda+i\ \sin\alpha) - (2\lambda+e^{i\alpha})\}b_k|^2$$

$$(5.3.3)\ |na_n+b_n e^{-i\alpha}|^2 \le |-(1-\tfrac{1}{\beta}) + \{\tfrac{\lambda+i\ \sin\alpha}{\beta} - (2\lambda+e^{i\alpha})\}e^{i\alpha}|^2 + \sum_{k=o}^{n-1}[|ka_k(1-\tfrac{1}{\beta}) + \{\tfrac{1}{\beta}(\lambda+i\ \sin\alpha)-(2\lambda+e^{i\alpha})\}b_k|^2 - |ka_k + b_k e^{-i\alpha}|^2]$$

Now we note that

$$(5.3.4)\quad |na_n+b_n e^{-i\alpha}|^2 = n^2|a_n|^2 + |b_n|^2 + 2\mathrm{Re}\{n\ e^{i\alpha}\ a_n\ \bar{b}_n\}$$

$$(5.3.5)\quad |-(1-\tfrac{1}{\beta})e^{-i\alpha} + \tfrac{\lambda+i\ \sin\alpha}{\beta} - 2\lambda - e^{i\alpha}|^2 = (2-\tfrac{1}{\beta})^2(\lambda+\cos\alpha)^2$$

$$(5.3.6)\quad |k\ a_k(1-\tfrac{1}{\beta}) + \{\tfrac{\lambda+i\ \sin\alpha}{\beta} - (2\lambda+e^{i\alpha})\}b_k|^2 = k^2|a_k|^2(1-\tfrac{1}{\beta})^2 + |b_k|^2\{\tfrac{\lambda^2+\sin^2\alpha}{\beta^2} - 2[\tfrac{2\lambda^2+\lambda\cos\alpha+\sin^2\alpha}{\beta}] + 4\lambda^2 + 4\lambda\cos\alpha + 1\} + 2\mathrm{Re}\{ka_k\bar{b}_k(1-\tfrac{1}{\beta})[\tfrac{\lambda-i\sin\alpha}{\beta} - (2\lambda+e^{-i\alpha})]\}\}$$

and

$$(5.3.7)\quad |ka_k+b_k e^{-i\alpha}|^2 = k^2|a_k|^2 + |b_k|^2 + 2\ \mathrm{Re}\{k\ a_k\ \bar{b}_k\ e^{i\alpha}\}$$

Hence from (5.3.6) and (5.3.7) we have

$$(5.3.8)\ \{|ka_k(1-\tfrac{1}{\beta}) + \{\tfrac{\lambda+i\ \sin\alpha}{\beta} - (2\lambda+e^{i\alpha})\}b_k|^2 - |ka_k+b_k e^{-i\alpha}|^2\} = -k^2|a_k|^2\,\tfrac{1}{\beta}(2-\tfrac{1}{\beta}) + |b_k|^2\{\tfrac{\lambda^2+\sin^2\alpha}{\beta^2} - 2(\tfrac{2\lambda^2+\lambda\cos\alpha+\sin^2\alpha}{\beta}) + 4\lambda(\lambda+\cos\alpha)\} + 2\mathrm{Re}\{ka_k\bar{b}_k[(\tfrac{\lambda-i\sin\alpha}{\beta})(1-\tfrac{1}{\beta}) + \tfrac{2\lambda+e^{-i\alpha}}{\beta} - 2(\lambda+\cos\alpha)]\}.$$

From (5.3.3), (5.3.4) and (5.3.8) we have

$$n^2|a_n|^2+|b_n|^2+2\,\mathrm{Re}\{ne^{i\alpha}a_n\bar{b}_n\}\le\left(2-\frac{1}{\beta}\right)^2(\lambda+\cos\alpha)^2+\sum_{k=0}^{n-1}\Big[-k^2|a_k|^2\,\frac{1}{\beta}\left(2-\frac{1}{\beta}\right)+|b_k|^2\,\Big\{\frac{\lambda^2+\sin^2\alpha}{\beta^2}-2\left(\frac{2\lambda^2+\lambda\cos\alpha+\sin^2\alpha}{\beta}\right)+4\,\lambda(\lambda+\cos\alpha)+2\mathrm{Re}\Big\{ka_k\bar{b}_k\Big[\frac{\lambda-i\sin\alpha}{\beta}\left(1-\frac{1}{\beta}\right)+\frac{2\lambda+e^{-i\alpha}}{\beta}-2(\lambda+\cos\alpha)\Big]\Big\}\Big]$$

This implies the result of the theorem.

It is wellknown [39] that every normalized meromorphic univalent function satisfies (5.3.1), hence theorem 2 yield the following :

Corollary 1. *If* $f \in B(\beta,\lambda,\sigma)$ *with respect to* F *and if* f *is univalent in* D, *then the coefficients of* f *are related by* (5.3.2).

Since F is starlike meromorphic univalent function of order σ , it is known [49] that

Theorem [Pommerenke] : *If* $F \in \Sigma_\sigma^*$, then

$$\sum_{k=0}^{\infty}(k+\sigma)|b_k|^2 \le (1-\sigma). \tag{5.3.9}$$

By a direct application of (5.3.9) it follows that

$$\sum_{k=0}^{n-1} k|a_k\, b_k| \le \Big(\sum_{k=0}^{n-1} k|a_k|^2\Big)^{1/2}\Big(\sum_{k=0}^{n-1} k|b_k|^2\Big)^{1/2} \le (1-\sigma)^{1/2} \tag{5.3.10}$$

Since f is univalent and so (5.3.1)holds, hence by using these estimates we get the following:

Corollary 2. *If* $f(z) \in B(\beta,\lambda,\sigma)$ *and if* f *is univalent then*

$$(n|a_n|+|b_n|) \leq \sqrt{(2-\tfrac{1}{\beta})^2(1-\lambda)^2+4\sqrt{1-\sigma}}$$

We deduce the following theorem of Libera by making $\beta\to\infty$ in corollary 2.

Theorem [Libera]: If $f(z) \in B(\lambda,\sigma)$ with respect to F and (5.3.1) is satisfied then

$$(n|a_n| + |b_n|) \leq 2\sqrt{(1-\lambda)^{1/2} + (1-\sigma)}$$

5.4. In this section we shall extend the definition of the meromorphic functions which are close-to-convex in some fixed direction.

Definition 4. If $f(z)$ is given by (5.1.1) and if for each $r (0 < r < 1)$ the image curve Γ_r, corresponding to $|z| = r$ through the function f, has the property that each straight line parallel to some fixed direction cuts Γ_r in atmost two points, then $f(z)$ is said to be convex in that direction.

It is to be noted that if $f(z)$ is convex in one direction it need not be univalent in $0 < |z| < 1$. An example to this effect has been given by Libera and Robertson [28]. They have further shown that if $f(z)$ is convex in the direction of imaginary axis then

$$\mathrm{Re}\{\frac{-z^2f'(z)}{1-z^2}\} > 0 \quad (|z| < 1)$$

Definition 5. We shall call $f(z) = \frac{1}{z} + a_o + a_1z + \ldots$, to be convex in the imaginary direction, of order λ, if

$$\text{(5.3.11)} \qquad \mathrm{Re}\{\frac{-z^2f'(z)}{1-z^2}\} > \lambda \quad (0 < \lambda \leq 1)$$

and denote the class of functions $f(z)$ characterized by (5.3.11), by $K(\lambda)$.

Definition 6. We shall denote the family of functions $f(z)$ having the representation (5.1.3) and satisfying the following condition

$$\left| \frac{\dfrac{-z^2 f'(z)}{1-z^2} - \lambda}{1-\lambda} - \beta \right| < \beta$$

by $K_\beta(\lambda)$.

$$\text{If} \quad P(z) = \left(\frac{-z^2 f'(z)}{1-z^2} - \lambda\right)\left(\frac{1}{1-\lambda}\right)$$

then $P(0) = 1$ and $\mathrm{Re}\,\{P(z)\} > 0$ for $z \in D$. Hence by lemma 1 we have

$$\frac{-z^2 f'(z)}{1-z^2} - \lambda = \frac{(1-\lambda)(1+\omega(z))}{1+\left(\frac{1-\beta}{\beta}\right)\omega(z)} .$$

If $F(z) = -\left(\dfrac{z}{1-z^2}\right)^{-1} = -\dfrac{1}{z} + z$ then

$$-\frac{zF'(z)}{F(z)} = \left(\frac{1+z^2}{1-z^2}\right) \quad \text{and} \quad \mathrm{Re}\left\{\frac{-zF'(z)}{F(z)}\right\} > 0.$$

Hence $F(z)$ satisfies the conditions of theorem 2 with $\alpha = \pi$, $\sigma = 0$, $b_o = 0$, $b_1 = 1$ and $b_n = 0$ for $n \geq 2$. Hence from theorem 2,we deduce the following theorem.

Theorem 3. *If* $f(z) \in K_\beta(\lambda)$, *is real on the real axis and*

$$\sum_{k=1}^{\infty} k|a_k|^2 \leq 1$$

then

$$|a_1 - 1| \leq (1-\lambda)\ (2 - \frac{1}{\beta})$$

and, for $n \geq 2$, we have

$$n^2|a_n|^2 \leq (1-\lambda)^2(2-\frac{1}{\beta})^2 - \sum_{k=o}^{n-1} k^2|a_k|^2 \frac{1}{\beta}\ (2 - \frac{1}{\beta}) + \frac{\lambda^2}{\beta^2} -$$

$$2(\frac{2\lambda^2}{\beta} - \lambda) - 4\lambda(1-\lambda)+2\ \mathrm{Re}\{a_1\ [\frac{\lambda}{\beta}\ (1- \frac{1}{\beta}) + \frac{2\lambda -1}{\beta} + 2(1-\lambda)]\}$$

Making $\beta \to \infty$ in theorem 3 the following theorem of Libera and Robertson [28] is obtained.

Theorem [Libera-Robertson] : If $f(z)$ be meromorphically convex in the direction of imaginary axis in $0 < |z| < 1$ and real on the real axis. Then

$$|a_n| \leq \frac{3(1+|a_1|)^{1/2}}{n} \leq \frac{4}{n} \text{ for } n \geq 2$$

and

$$|a_1| \leq 3.$$

If $f(z)$ is univalent, then

$$|a_n| \leq \frac{2\sqrt{2}}{n}, |a_1| \leq 1.$$

CHAPTER 6

Some Growth Estimates For Typically Real Functions of Class T(p)

6.1. The function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, regular in $|z| < 1$, is called typically real if it is real on the diameter $-1 < z < 1$ and at other points of the disc $|z| < 1$ it has the property that $Im\{f(z)\}$ and $Im\{z\}$ always have the same sign. Such a class of typically real functions was introduced by Rogosinski [58] and is called the T(1) class. The integral representation of such a class of functions has been obtained by M.S. Robertson [53] and also by Golusin [16]. Recently, W.E. Kirwan [26] has obtained the necessary and sufficient condition on the measure function occuring in the integral ([53] and [16]) with the growth of $M(r,f) = \max_{|z|=r} |f(z)|$ for the functions of class T(1). In the present chapter our aim is to extend the results of W.E. Kirwan to the subclass of typically real functions of class T(p) introduced by M.S. Robertson [54]. Gelfer [13] defined the class T(p) by $f(z) = \sum_{n=1}^{\infty} c_n z^n$ such that

(i) $f(z)$ is regular in the disc $|z| < 1$ and all the coefficients are real.

(ii) there is a number $\delta = \delta(f)$, $0 < \delta < 1$, such that for each r in the interval $1-\delta < r < 1$, $Im\{f(z)\}$ changes sign 2p-times on the circle $|z|=r$.

If $p = 1$, the class T(p) reduces to the class T(1). The T(p) class in particular includes all p-valent functions with real coefficients and starlike with respect to a point on the real axis. Gelfer [13] has shown that if $f(z)$ belongs to the class T(p) then it can be expressed as following :

where A_o^m are real coefficients expressed by the expressions given in article 6.2, and the numbers $\theta_m, 0 \leq \theta_m \leq \pi$ $(m=1,2,\ldots,p-1)$ denote the change of sign of $\mathrm{Im}\{f(z)\}$ on $|z| = r$ and $\mu(\theta)$ is an increasing function such that $\frac{1}{\pi}\int_0^\pi d\mu(\theta) = 1$. The function $S(z, \cos\theta)$ is defined in (6.2.2). For simplicity, we shall denote the sum and product on the right hand side of (6.1.1) by $B(z)$ and $A(z)$ respectively. Note that (6.1.1) is also an inductive consequence of the results of Goodman and Robertson [18]. Throughout this chapter we shall restrict ourselves to functions belonging to the class $T(p)$, having the representation (6.1.1). We shall denote by $T^*(p)$ the subclass of functions of class $T(p)$ which satisfy the additional condition

$$(6.1.2) \qquad \limsup_{r \to 1} M\left(r, \prod_{k=1}^{p-1} S(z, \cos\theta_k)\right)(1-r)^{2(p-1)} = \alpha$$

where $S(z, \cos\theta_k)$ is the same as in (6.1.1) and α is strictly greater than zero. The results derived for the class $T^*(p)$ hold trivially for the class $T(1)$.

6.2. Notations and symbols : In this section we list the notations and symbols which we shall use and also specify two inequalities, (6.2.5) and (6.2.6), to which we shall refer frequently in the sequel.

$$(6.2.1) \qquad C_k^i = \frac{k(k-1)\ldots(k-i+1)}{i!}$$

$$(6.2.2) \qquad S(z, \cos\theta) = z(1 - 2z\cos\theta + z^2)^{-1}$$

$$(6.2.3) \qquad S_1(z, \cos\theta) = (1-z^2)(1-2z\cos\theta + z^2)^{-1}$$

$$A_o^k = \sum_{i=o}^{k} (-1)^{k-i} a_i \Delta_k^i \qquad (k = 1, 2, 3, \ldots, p-1)$$

$$\Delta_k^i = {}_{k-i}^{2i} C_{k-1}^{\frac{k-i-2}{2}} + {}_{k-i-2}^{2^2.2i} C_{k-3}^{\frac{k-i-4}{2}} \sum_{r_1>r_2=1}^{k-1} \cos\theta_{r_1} \cos\theta_{r_2} + \ldots$$

$$\ldots + 2^{k-i-2} \frac{2i}{2} \sum_{r_1>\ldots>r_{k-i-2}=1}^{k-1} \cos\theta_{r_1} \cos\theta_{r_2} \ldots \cos\theta_{r_{k-i-2}} +$$

$$+ 2^{k-i} \sum_{r_1>\ldots>r_{k-i}=1}^{k-1} \cos\theta_{r_1} \ldots \cos\theta_{r_{k-i}}$$

(6.2.4) <u>if</u> k-i > 0 and <u>is</u> even ,

$$\Delta_k^i = 2.{}_{k-i-1}^{2i} C_{k-2}^{\frac{k-i-3}{2}} \sum_{r_1=1}^{k-1} \cos\theta_{r_1} + 2^3.{}_{k-i-3}^{2i} C_{k-4}^{\frac{k-i-5}{2}} \sum_{r_1>r_2>r_3=1}^{k-1} \cos\theta_{r_1} \cos\theta_{r_2} \cos\theta_{r_3}$$

$$+ \ldots + 2^{k-i} \sum_{r_1>r_2>\ldots>r_{k-i}=1}^{k-1} \cos\theta_{r_1} \cos\theta_{r_2} \ldots \cos\theta_{r_{k-i}}$$

<u>if</u> k - i > 1 and <u>is</u> odd ,

$$\Delta_k^k = 1.$$

(6.2.5) $$|\mu(\theta) - \mu(s)| \leq A\, |\theta+s|\, |\theta-s|^{1-\lambda}$$

(6.2.6) $$|\mu(\theta) - \mu(s)| \leq B\, |\theta+s-2\pi|\, |\theta-s|^{1-\lambda}$$

where s and θ lie in the interval $[0,\pi]$, λ is a real number and A and B are constants.

(6.2.7) $$A(r) = O\{(1-r)^{-\lambda-2(p-1)}\}$$

$$(6.2.8)\qquad B(r) = \sum_{m=1}^{p-1}\sum_{k=1}^{m}\sum_{n=m}^{\infty} r^n \cdot \frac{|a_k|.2k.n(n^2-1^2)\dots(n^2-(m-1)^2)}{(m+k)!(m-k)!}$$

(6.2.9) $a_1(n)$ denote the coefficient of z^n in $A(z)$.

6.3. In this section we shall state lemmas which we shall need.

Lemma 1. <u>If</u> $f(z) = \int_0^\pi S(z,\cos\theta)d\mu(\theta)$ <u>and</u> $0 < \lambda \leq 2$, then

$$M(r,f) = \max_{|z|=r} |f(z)| \leq 0(1-r)^{-\lambda}$$

<u>if</u> and only <u>if</u>, $\mu(\theta)$ satisfies (6.2.5) and (6.2.6).

This lemma is due to W.E. Kirwan ([26]; p.9). Hence we omit the proof.

Lemma 2. We find that $|A(z)| \leq A(r)$

Proof ; For $z = re^{i\theta}$, we have

$$(6.3.1)\qquad |S(z,\cos\theta_i)| = |z(1-2z\cos\theta_i + z^2)^{-1}|$$

$$= [|\tfrac{1+z^2}{z}|-4\,\mathrm{Re}(\tfrac{1+z^2}{z})\cos\theta_i + 4\cos^2\theta_i]^{-\frac{1}{2}}$$

The expression on the right hand side of (6.3.1) attains its maximum at $\theta_i = 0$ when $\mathrm{Re}(\frac{1+z^2}{z}) \geq 2$, at $\theta_i = \pi$ when $\mathrm{Re}(\frac{1+z^2}{z}) \leq -2$ and at $\theta_i = \cos^{-1}(\mathrm{Re}(\frac{1+z^2}{2z}))$ when $|\mathrm{Re}(\frac{1+z^2}{z})| \leq 2$.

From the above facts we obtain that

$$|S(re^{i\theta}, \pm 1)| \leq \frac{r}{(1-r)^2}, \text{ if } |\mathrm{Re}(\tfrac{1+z^2}{z})| \geq 2$$

and

$$|S(re^{i\theta},\cos^{-1}(\mathrm{Re}(\tfrac{1+z^2}{2z})))| \leq \frac{1}{2}\left(\frac{r}{1-r}\right)^2, \text{ if } |\mathrm{Re}(\tfrac{1+z^2}{z})| < 2$$

Hence we obtain

$$|A(re^{i\theta})| \leq |\prod_{k=1}^{p-1} S(re^{i\theta},\cos\theta_k)||\int_0^{\pi} S(re^{i\theta}, \cos\phi)d\mu(\phi)|$$

$$\leq O[(1-r)^{-2(p-1)}]|\int_0^{\pi} S(re^{i\theta}, \cos\phi)d\mu(\phi)|$$

$$\leq O[(1-r)^{-\lambda-2(p-1)}] \equiv A(r).$$

The last inequality is obtained by lemma 1.

Lemma 3: We find that $|B(z)| \leq B(r)$

Proof : Clearly, with $z = re^{i\theta}$, we have

$$|B(z)| \leq \sum_{m=1}^{p-1} |A_o^m| \prod_{k=1}^{m} |S(z,\cos\theta_k)|$$

$$\leq \sum_{m=1}^{p-1} \left[\sum_{k=1}^{m} \frac{|a_k|}{(m+k)!}\frac{2k(2m-1)!}{(m-k)!}\right]\left[\sum_{n=m}^{\infty} \frac{n(n^2-1^2)\dots(n^2-(m-1)^2)}{(2m-1)!} r^n\right]$$

$$\equiv B(r).$$

6.4. In this section we shall prove the following theorems :

Theorem 1. If $f(z) = z^2 + \sum_{n=3}^{\infty} a_n z^n$ belongs to the class $T(2)$ and $(\mathrm{Im}\{f(z)\})^{1/2}$ and $\mathrm{Im}\{z\}$ have the same sign, then

if and *only if*, (6.2.5) and (6.2.6) hold with $\lambda = 2$.

Proof : Sufficiency part of theorem 1 follows from lemma 2. In order to establish the necessity part, we observe that $\{f(z)\}^{1/2}$ is analytic in the unit disc and belongs to the class $T(1)$. Thus, if conditions (6.2.5) and (6.2.6) do not hold then by lemma 1, there exists a sequence $\{r_n e^{i\psi_n}\}$ on which

$$|f(r_n e^{i\psi_n})|^{1/2} (1-r_n)^2 \to \infty$$

as $n \to \infty$ and $r_n \to 1$.

Hence on the sequence $\{r_n e^{i\psi_n}\}$

$$M(r_n, f) \neq O(1-r_n)^4$$

whenever conditions (6.2.5) and (6.2.6) are violated.

This completes the proof.

In general, we have

Theorem 2. *If* $f(z) \in T^*(p)$ *and* $0 < \lambda \leq 2$ then

$$M(r,f) = \max_{|z|=r} |f(z)| = A(r) + B(r)$$

if (6.2.5) *and* (6.2.6) *are satisfied.*

Conversely, if $f(z) \in T^*(p)$ *and* $0 < \lambda \leq 2$ *and* $M(r,f) = A(r) + B(r)$, then (6.2.5) *and* (6.2.6) *hold for every sequence* $\{s_n\}$ and $\{\theta_n\}$ *in each of the following cases* :

(a) *sequences* $\{s_n\}$ *and* $\{\theta_n\}$ *do not have limit point zero.*

(b) $\theta_0 = 0$ *or* π *and* $\lambda \geq 1$.

Proof : The sufficiency part of theorem 2 follows immediately from lemmas 1, 2 and 3.

In order to establish the necessity part, we shall show that, if either of two conditions (6.2.5) or (6.2.6) fails then $M(r,f) \neq A(r) + B(r)$. Choose the sequences $\{s_n\}$, $\{t_n\}$ and $\{A_n\}$ such that $\lim_{n\to\infty} s_n, t_n, A_n = s', t', \infty$ respectively and $0 < s', t' < \pi$ and $|\mu(s_n) - \mu(t_n)| > A_n|s_n+t_n||s_n-t_n|^{1-\lambda}$ to violate condition (6.2.5). Without loss of generality, we may assume that $s_n \geq t_n$ and since $\mu(\theta)$ is bounded, so $s'=t'$. Further, suppose that $1-r_n=s_n-t_n$ and $|z_n| = r_n$. We shall show that for the sequence $\{z_n\} = \{r_n e^{i\theta_o}\}$

$\lim_{n\to\infty}|F(z_n)|(1-r_n)^{\lambda+2(p-1)} \to\infty$ where $F(z_n)=A(z_n)$ and θ_o is an element in $[0,2\pi]$ for which (6.1.2) holds. We shall consider the following three cases separately (i) $0 \leq \theta_o < \pi$, (ii) $\theta_o = \pi$ and (iii) $\pi < \theta_o < 2\pi$. In case (i) we have

$$|F(z_n)| = \frac{1}{\pi} \prod_{k=1}^{p-1} |S(z_n,\cos\theta_k)||\int_0^\pi S(z_n,\cos\theta)d\mu(\theta)|$$

$$= \left|\frac{z_n}{\pi(1-z_n^2)}\right| \prod_{k=1}^{p-1} |S(z_n,\cos\theta_k)||\int_0^\pi S_1(z_n,\cos\theta)d\mu(\theta)|$$

Since $\mathrm{Re}\,\{S_1(z_n,\cos\theta)\} > 0$, we have

$$|F(z_n)| \geq \frac{A}{|1-r_n^2e^{2i\theta_o}|}\left|\prod_{k=1}^{p-1} S(z_n,\cos\theta_k)\right|\int_0^\pi \mathrm{Re}\{S_1(r_ne^{i\theta_o}, \cos\theta)\}\, d\mu(\theta)$$

$$\geq \frac{A}{|1-r_n^2e^{2i\theta_o}|}\left|\prod_{k=1}^{p-1} S(z_n,\cos\theta_k)\right| \mathrm{Re}\{S_1(r_ne^{i\theta_o},\cos x_n)\}\int_{\theta_o}^{\theta_o+s_n-t_n} d\mu(\theta)$$

$$= \frac{A\left|\prod_{k=1}^{p-1} S(r_n e^{i\theta_o},\cos\theta_k)\right|(1-r_n^2)(1-2r_n\cos x_n\cos\theta_o+r_n^2)\{\mu(\theta_o+s_n-t_n)-\mu(\theta_o)\}}{|1-r_n^2 e^{2i\theta_o}|(1-2r_n\cos(\theta_o-x_n)+r_n^2)(1-2r_n\cos(\theta_o+x_n)+r_n^2)}$$

for some x_n lying in the interval $[\theta_o,\theta_o+s_n-t_n]$ and for some suitable constant $A(>0)$. Since

$$\frac{1-2r_n\cos x_n\cos\theta_o+r_n^2}{1-2r_n\cos(x_n+\theta_o)+r_n^2} = 1 - \frac{2r_n\sin x_n\sin\theta_o}{(1-r_n)^2+4r_n\sin^2\left(\frac{x_n+\theta_o}{2}\right)}$$

$$> 1 - \frac{\sin x_n\sin\theta_o}{2\sin^2\left(\frac{x_n+\theta_o}{2}\right)} \geq \frac{1}{2}.$$

and

$$1-2r_n\cos(\theta_o-x_n)+r_n^2 = (1-r_n)^2+4r_n\sin^2\left(\frac{x_n-\theta_o}{2}\right)$$

$$= (1-r_n)^2+4r_n\left(\frac{x_n-\theta_o}{2}\right)^2 - \ldots$$

$$\leq 2(1-r_n)^2$$

Further, since $\mu(\theta)$ has its finite upper derivative, therefore for some suitable positive number ℓ_1, $0 \leq \ell_1 < \infty$, we have

$$(6.4.1)\quad |F(z_n)| \geq \frac{\left|\prod_{k=1}^{p-1} S(r_n e^{i\theta_o},\cos\theta_k)\right|(1-r_n^2)\{A_n(1-r_n)^{1-\lambda}(s_n+t_n)-\ell_1(1-r_n)\}}{2(1-r_n)^2|1-r_n^2 e^{2i\theta_o}|}$$

Case (ii), if $\theta_o = \pi$, then proceeding as above, we obtain

$$|F(z_n)| \geq \frac{A\left|\prod_{k=1}^{p-1} S(r_n e^{i\pi}, \cos\theta_k)\right|(1-r_n^2)\{\mu(\pi)-\mu(\pi-s_n+t_n)\}}{2(1-r_n)^2\ (1+2r_n\cos x_n + r_n^2)}$$

where $\pi-(s_n-t_n) \leq x_n \leq \pi$. Again from the fact

$$1 + 2r_n\cos x_n + r_n^2 \leq 1+r_n^2 +2r_n \cos\{\pi-(s_n-t_n)\}$$

$$= 1 + r_n^2 - 2r_n\{1 - \frac{(1-r_n)^2}{2^2} + \frac{(1-r_n)^3}{6^3} - \ldots\}$$

$$\leq 1 + r_n^2 -2r_n + \frac{(1-r_n)^2}{2} r_n$$

$$\leq \frac{3}{2}(1-r_n)^2$$

Hence, again, for some suitable constant ℓ_2, $0 \leq \ell_2 < \infty$, we have

$$(6.4.2)\ |F(z_n)| \geq \frac{\left|2A \prod_{k=1}^{p-1} S(r_n e^{i\pi}, \cos\theta_k)\right|(1-r_n^2)\{A_n(1-r_n)^{1-\lambda}(s_n+t_n)-\ell_2(1-r_n)\}}{3(1-r_n)^2\ (1-r_n^2)}$$

In case (iii) when $\pi < \theta_o < 2\pi$, we have

$$|F(z_n)| \geq \frac{A\prod_{k=1}^{p-1}\left|S(r_n e^{i\theta_o}, \cos\theta_k)\right|(1-r_n^2)(1-2r_n\cos\phi_o\cos x_n + r_n^2)\int_{\phi_o}^{\phi_o+s_n-t_n} d\mu(\theta)}{\left|1-r_n^2 e^{-2i\phi_o}\right|(1-2r_n\cos(x_n-\phi_o)+r_n^2)(1-2r_n\cos(x_n+\phi_o)+r_n^2)}$$

where $\phi_o = 2\pi - \theta_o$. This evidently implies (6.4.1). Now from (6.4.1) and (6.4.2) and the fact that $f(z)$ is assumed to satisfy (6.1.2) implies that

$$(6.4.3)\quad (1-r_n)^{\lambda+2(p-1)}|F(z_n)| \geq \frac{A(\alpha-\varepsilon)(1-r_n^2)(1-r_n)^{\lambda}\{A_n(s_n+t_n)-\ell(1-r_n)^{\lambda}\}(1-r_n)^{1-\lambda}}{(1-r_n)^2|1-r_n^2 e^{2i\theta_o}|}$$

where ℓ is a suitable constant.

Evidently, it follows from (6.4.3) that if $\{s_n\}$ and $\{t_n\}$ do not have limit point zero then $M(r,f) \neq A(r) + B(r)$. When $\theta_o = 0$, or π, then

$$(6.4.4)\quad (1-r_n)^{\lambda+2(p-1)}|F(z_n)| \geq \frac{A(\alpha-\varepsilon)}{(1-r_n)}\{A_n(s_n+t_n) - \ell(1-r_n)^{\lambda}\}$$

$$\geq \frac{A(\alpha-\varepsilon)(1-r_n)}{(1-r_n)}\{A_n - \ell(1-r_n)^{\lambda-1}\}$$

$$= A(\alpha-\varepsilon)\{A_n - \ell(1-r_n)^{\lambda-1}\}$$

$$\to \infty \text{ as } n \to \infty \text{ if } \lambda \geq 1.$$

This proves that (6.2.5) is necessary in each of the cases (a) and (b). Similarly, it can be demonstrated that (6.2.6) is also necessary in each of the cases (a) and (b). This completes the proof of theorem 2.

A function $\mu(\theta)$ defined on $[0,\pi]$ is said to satisfy a Lipschitz condition of order $\lambda (\lambda \geq 0)$ at $t = \phi$ on $[0,\pi]$, (denoted by $\mu(\theta) \in \mathrm{Lip}(\lambda)$ at $t = \phi$), if for all $\theta \in [0,\pi]$, we have $|\mu(\theta) - \mu(\phi)| \leq A(\theta)|\theta - \phi|^{\lambda}$. We assume for our purposes that $A(\theta) \neq 0$ for all $\theta \in [0,\pi]$.

Theorem 3. <u>If</u> $f(z) \in T^*(p)$ and <u>if</u> $\theta_o \neq 0, \pi$ and $0 < \lambda < 1$ then

$$|f(re^{i\theta_o})| = O\{A(r)\} \quad , \quad \text{as } r \to 1$$

<u>if</u> and <u>only</u> <u>if</u> , $\mu(\theta) \in$ <u>Lip</u> $(1-\lambda)$ <u>at</u> $\theta = \theta_o$.

If $\theta_o = 0$ and π and $0 < \lambda \leq 2$, then

$$|f(re^{i\theta_o})| = O\{A(r)\} \text{ as } r \to 1$$

if and only if, $\mu(\theta) \in \mathrm{Lip}\,(2-\lambda)$ at $\theta_o = 0$ and π, respectively.

Proof : By definition of $A(z)$, we have for $z = re^{i\theta_o}$

$$|A(z)| \leq A(1-r)^{-2(p-1)} \left|\int_o^\pi S(z\ ,\cos\theta)d\mu(\theta)\right|$$

Now let

$$I = \int_o^\pi S(re^{i\theta_o}, \cos\theta)\, d\mu(\theta)$$

We restrict ourselves to the case when θ_o lies in the interval $(0,\frac{\pi}{2})$. The other part when θ_o lies in the interval $[\frac{\pi}{2},\pi)$ can be treated similarly. The case when $\pi < \theta_o < 2\pi$, is diametrically symmetric, hence we omit its reference in the proof. Integrating I by parts we have

$$I=S(re^{i\theta_o},-1)[\mu(\pi)-\mu(\theta_o)]+S(re^{i\theta_o},1)[\mu(\theta)]_o^{\theta_o}-\int_o^\pi[\mu(\theta)-\mu(\theta_o)][\tfrac{d}{d\theta}S(re^{i\theta_o},\cos\theta)]d\theta$$

$$=O(1)+\Big(\int_o^{\theta_o-(1-r)}+\int_{\theta-(1-r)}^{\theta_o+(1-r)}+\int_{\theta_o+(1-r)}^{\pi}\Big)[\mu(\theta)-\mu(\theta_o)][\tfrac{d}{d\theta}S(re^{i\theta_o},\cos\theta)]d\theta$$

$$= O(1) + I_1 + I_2 + I_3$$

Now,

$$|I_j| \leq A\int \qquad |\mu(\theta)-\mu(\theta_o)|\,|\theta-\theta_o|^{-2}\, d\theta \leq A \int \qquad (\theta_o-\theta)^{-(1+\lambda)}d\theta$$

$$= O(1-r)^{-\lambda} \text{ for } j = 1,3.$$

But, for I_2, we have

$$|I_2| \leq \frac{A}{(1-r)^2}\int_{\theta_o-(1-r)}^{\theta_o+(1-r)} |\theta-\theta_o|^{1-\lambda} d\theta \leq A(1-r)^{-\lambda}$$

This completes the sufficiency part of the theorem.

The case when $\theta_o = 0$ (respectively π) and $\mu(\theta) \in \text{Lip}\ (2-\lambda)$ can be treated similarly.

Now we shall establish the necessity part of theorem 3 when $\theta_o = 0$ (respectively π). Following exactly the similar lines as that of theorem 2, we have

$$|A(z_n)| \geq \frac{A \prod_{k=1}^{p-1} |S(re^{i\theta_o}, \cos\theta_k)(1-r_n^2)J|}{4(1-r_n^2)\ (1-r_n)^2}$$

where

$$J = \begin{cases} \int_{s_n}^{\pi} d\mu(\theta) & ;\text{if } \theta_o = \pi\ ,\ \lim_{n\to\infty} s_n = \pi\ \ ,\ s_n \leq \pi\ . \\ \int_{o}^{s_n} d\mu(\theta) & ;\text{if } \theta_o = 0\ ,\ \lim_{n\to\infty} s_n = 0,\ \ s_n \geq 0\ . \end{cases}$$

Hence,

$$J = \begin{cases} \mu(\pi) - \mu(s_n) & ;\text{if } \theta_o = \pi,\ \lim_{n\to\infty} s_n = \pi \\ \mu(s_n) - \mu(0) & ;\text{if } \theta_o = 0,\ \lim_{n\to\infty} s_n = 0 \end{cases}$$

Hence if $\mu(\theta) \notin \text{Lip}\ (2-\lambda)$ at θ_o, then

$$|\mu(\theta_o) - \mu(s_n)| > A_n(\theta)|\theta_o - s_n|^{2-\lambda} = A_n(\theta)\ (1-r_n)^{2-\lambda}$$

and

$$|A(z_n)| \geq \frac{A\, A_n \left| \prod_{k=1}^{p-1} S(re^{i\theta_o}, \cos\theta_k) \right| (1-r_n^2)(1-r_n)^{2-\lambda}}{4(1-r_n)^3(1+r_n)}$$

Hence,

$$(1-r_n)^{\lambda+2(p-1)} |A(z_n)| \to \infty \text{ as } n \to \infty, \text{if } \mu(\theta) \notin \text{Lip } (2-\lambda) \text{ at } \theta_o.$$

This proves the necessity part of theorem 3 in the case when $\theta_o = 0$ (respectively π). Similarly, if $\theta_o \neq 0, \pi$, then

$$|A(z_n)| \geq \frac{A \left| \prod_{k=1}^{p-1} S(r_n e^{i\theta_o}, \cos\theta_k) \right| (1-r_n^2)\{A_n (1-r_n)^{1-\lambda}\}}{2(1-r_n)^2 \left|1-r_n^2 e^{2i\theta_o}\right|}$$

and this implies that

$$(1-r_n)^{\lambda+2(p-1)} |A(z_n)| \to \infty \quad \text{as } n \to \infty$$

if $\mu(\theta) \notin \text{Lip } (1-\lambda)$ at $\theta = \theta_o$. This completes the proof of theorem 3.

Theorem 4. <u>If</u> $f(z) \in T^*(p)$ and $f(\pm r) = O\{(1-r)^{-\lambda-2(p-1)}\}$ <u>as</u> $r \to 1$, then

$$L(r,f) = \frac{1}{2\pi} \int_0^{2\pi} r|f'(re^{i\theta})|\, d\theta$$

$$= \begin{cases} \dfrac{2\pi p A r^{p-1}(Bp+C)}{(1-r)^{\lambda+2(p-1)}} + \displaystyle\sum_{m=1}^{p-1} \frac{|2\pi A_o^m| m r^{m-1}}{(1-r)^{2m+1}} & ; \text{ if } 1 < \lambda \leq 2 \\[2ex] \dfrac{2\pi p A r^{p-1}}{(1-r)^{2p-1}} \{B + C \log (1-r)^{-1}\} + \displaystyle\sum_{m=1}^{p-1} \frac{2\pi p^2 |A_o^m| m r^{m-1}}{(1-r)^{2m+1}} & ; \text{ if } \lambda = 1 \end{cases}$$

where A, B, C, D, E are absolute constants.

Proof of Theorem 4: We note that if $\theta \neq 0$, or π, then for $z = re^{i\phi}$

$$(6.4.5) \qquad \int_0^{2\pi} |S'(z,\cos\theta)|\,d\phi \leq A(1-r)^{-1}|\operatorname{cosec}\theta|$$

and

$$|S'(z,\cos\theta)| \leq 2A(1-r)^{-2}\,|\operatorname{cosec}\theta|$$

Also for $0 \leq \theta < 2\pi$, we have

$$|f'(re^{i\theta})| \leq \begin{cases} \dfrac{pr^{p-1}(1+r^2+2r\cos\theta)^{1/2}}{(1+r^2-2r\cos\theta)^{\frac{2p+1}{2}}} & ;\ \text{if } \cos\theta \geq \dfrac{2r}{1+r^2} \\[2ex] \dfrac{pr^{p-1}(1+r^2-2r\cos\theta)^{1/2}}{(1+r^2+2r\cos\theta)^{\frac{2p+1}{2}}} & ;\ \text{if } \cos\theta \leq \dfrac{-2r}{1+r^2} \\[2ex] \dfrac{pr^{p-1}(1+r^4-2r^2\cos 2\theta)^{1/2}}{|\sin\theta|^{p+1}\,(1-r^2)^{p+1}} & ;\ |\cos\theta| < \dfrac{2r}{1+r^2} \end{cases}$$

Thus

$$(6.4.6) \qquad \int_0^{2\pi} |f'(re^{i\theta})|\,d\theta = \left(\int_{E_1} + \int_{E_2} + \int_{E_3}\right)|f'(re^{i\theta})|\,d\theta$$

where the sets E_1, E_2, E_3 are chosen in such a way that in $E_1, \cos\theta \geq \frac{2r}{1+r^2}$; in E_2, $\cos\theta \leq \frac{-2r}{1+r^2}$ and in E_3, $|\cos\theta| \leq \frac{2r}{1+r^2}$ and E_1 E_2 $E_3 = [0,2\pi]$.

If all a_k $(k = 1,2,3,\dots,p-1)$ are zero, then

$$\int_0^{2\pi} |f'(re^{i\theta})|d\theta \le pr^{p-1}\Big[\int_{E_1} \frac{(1+r^2+2r\cos\theta)^{\frac{1}{2}}}{(1+r^2-2r\cos\theta)^{\frac{2p+1}{2}}}d\theta + \int_{E_2} \frac{(1+r^2-2r\cos\theta)^{\frac{1}{2}}}{(1+r^2+2r\cos\theta)^{\frac{2p+1}{2}}}d\theta + \int_{E_3} \frac{(1+r^4-2r^2\cos 2\theta)^{1/2}}{|\sin\theta|^{p+1}(1-r^2)^{p+1}}d\theta\Big]$$

$$\le 2\pi p r^{p-1}\Big[\frac{1+r}{(1-r)^{2p+1}} + \frac{1+r}{(1-r)^{2p+1}} + \frac{1}{2\pi}\int_{E_3}\Big\{\frac{(1-r^2)^{-2p}}{|\sin\theta|^{2p+2}} + \frac{4r^2(1-r^2)^{-2p-2}}{|\sin\theta|^{2p}}\Big\}^{1/2}d\theta\Big]$$

$$\le 2\pi p r^{p-1}\Big[\frac{2(1+r)}{(1-r)^{2p+1}} + \frac{(1+r^2)^{p/2}(1+7r^2)^{1/2}}{(1-r)^{2p+1}}\Big] \equiv C(r).$$

If all $a_k (k = 1, 2, 3, \ldots, p-1)$ are not zero, then the factor

$$(6.4.7)\quad \int_0^{2\pi}\Big|\sum_{m=0}^{p-1} A_o^m\{\prod_{k=1}^{m} S(z,\cos\theta_k)\}'\Big|d\theta \le 2\sum_{m=1}^{p-1}\frac{|A_o^m|2\pi p^2 m r^{m-1}}{(1-r)^{2m+1}} \equiv D(r)$$

must be added to the right hand side of (6.4.6). For $\theta_i \ne 0, \pi$, define

$$H(r,\cos\theta_p) = \int_0^{2\pi}\Big|\sum_{i=1}^{p}\{\prod_{\substack{j\ne i\\ j=1}}^{p} S(re^{i\phi},x_j)\}S'(re^{i\phi},x_i)\Big|d\phi$$

where x_j stands for $\cos\theta_j$. Since $|S(re^{i\phi},x_k)| \le \frac{r}{(1-r)^2}$

we have

$$H(r,\cos\theta_p) \le \frac{r^{p-1}}{(1-r)^{2(p-1)}}\int_0^{2\pi}\sum_{i=1}^{p}|S'(re^{i\phi},x_i)|d\phi$$

Thus by using (6.4.5) and (6.4.6) we obtain

$$(6.4.8)\qquad H(r,\cos\theta_p) \leq \begin{cases} 2\pi r^{p-1}\sum_{i=1}^{p}|\operatorname{cosec}\theta_i|(1-r)^{-2p+1}; \\ \text{if } \theta_i \neq 0,\ \pi \\ C(r)\ ;\ \text{if } \theta_i = 0 \text{ or } \pi \end{cases}$$

If $f(z)$ belongs to the class $T^*(p)$ and is given by (6.1.1), then

$$L(r,f)=r\int_0^{2\pi}|f'(re^{i\phi})|d\phi \leq A\int_0^{\pi}H(r,\cos\theta_p)d\mu(\theta)+A\int_0^{2\pi}K(r,\phi)d\phi \equiv L + M$$

where $K(r,\phi)$ is given by

$$K(r,\phi) = \sum_{m=1}^{p-1}\left|A_0^m\left(\sum_{i=1}^{p}\prod_{\substack{k=1\\k\neq i}}^{m}S(z,\cos\theta_k)\,S'(z,\cos\theta_i)\right)\right|$$

If $f(\pm r) = O(1-r)^{-\lambda-2(p-1)}$ with $\theta_0 = 0$ and $\theta_0 = \pi$ respectively, then by theorem 3, $\mu(\theta)$ belongs to the class Lip $(2-\lambda)$ at $\theta_0 = 0$ and π respectively. Now,

$$L = \int_0^{\pi} H(r,\cos\theta_p)d\mu(\theta)=\left(\int_0^{1-r}+\int_{1-r}^{\pi/2}+\int_{\pi/2}^{\pi-(1-r)}+\int_{\pi-(1-r)}^{\pi}\right)H(r,\cos\theta_p)d\mu(\theta))$$

Using (6.4.8) we have

$$|I_1| \leq C(r)|\mu(1-r) - \mu(0)| = A(r)$$

Similarly by using (6.4.8) and the estimate $\operatorname{cosec}\theta_i \leq A/\theta_i$, we have

$$|I_2| \leq \frac{2p^2\pi r^{p-1}B}{(1-r)^{2p-1}}\left[\left[\frac{\mu(\theta)-\mu(0)}{\theta-0}\right]_{1-r}^{\pi/2}+\int_{1-r}^{\pi/2}\frac{\mu(\theta)-\mu(0)}{\theta^2}d\theta\right]$$

Since $|\mu(\theta) - \mu(0)| \leq A\theta^{2-\lambda}$, the preceding inequality implies that

$$|I_2| \leq \begin{cases} \dfrac{2\pi pAr^{p-1}(Bp+C)}{(1-r)^{\lambda+2p-2}}, & \text{if } 1 < \lambda \leq 2 \\[2ex] \dfrac{2\pi pAr^{p-1}}{(1-r)^{2p-1}} \{B + C \log (1-r)^{-1}\}, & \text{if } \lambda = 1 \\[2ex] \dfrac{2\pi p^2 r^{p-1}}{(1-r)^{2p-1}} \{D + E (1-r)^{1-\lambda}\}, & \text{if } 0 < \lambda < 1 \end{cases}$$

where A, B, C, D are absolute constants. Similarly estimating $|I_3|$ and $|I_4|$ and combining all the inequalities, the result of theorem 4, follows.

Corollary 1. <u>If</u> $f(z) = z^p + \sum_{m=1}^{\infty} a_{m+p} z^{m+p} \in T^*(p)$, <u>then</u>

$$L(r,f) = \begin{cases} O\{(1-r)^{-\lambda-2(p-1)}\}; & \text{if } 1 < \lambda \leq 2 \\[1ex] O\{(1-r)^{-(2p-1)} \log (\frac{1}{1-r})\}, & \text{if } \lambda = 1 \\[1ex] O\{(1-r)^{-2p+1}\}, & \text{if } 0 < \lambda < 1 \end{cases}$$

Corollary 2. <u>If</u> $f(z) = z^p + \sum_{m=1}^{\infty} a_{m+p} z^{m+p} \in T^*(p)$, <u>then</u>

$$|a_n| = \begin{cases} O\{n^{\lambda+2p-3}\}, & \text{if } 1 < \lambda \leq 2 \\[1ex] O\{n^{2p-2} \log n\}; & \text{if } \lambda = 1 \\[1ex] O\{n^{2p-2}\}; & \text{if } 0 < \lambda < 1 \end{cases}$$

<u>Proof</u> : We know that

$$n|a_n| \leq \frac{r^{-n+1}}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})| d\theta = r^{-n} L(r,f)$$

and so by corollary 1, we have

$$n|a_n| \leq \begin{cases} r^{-n} O\{(1-r)^{-\lambda-2(p-1)}\}, \text{ if } 1 < \lambda \leq 2 \\ r^{-n} O\{(1-r)^{-2p+1} \log (1-r)^{-1}\}; \text{ if } \lambda = 1 \\ r^{-n} O\{(1-r)^{-2p+1}\}; \text{ if } 0 < \lambda < 1 \end{cases}$$

Choosing $r = 1 - \frac{1}{n}$ we obtain the result of corollary 2.

Theorem 5 : If $f(z)$ belongs to the class $T(p)$ and $\mu(\theta)$ satisfies the conditions (6.2.5) and (6.2.6) with $\lambda = 0$, then

$$\lim_{m \to \infty} \frac{B^m(0)}{m!} = \lim_{m \to \infty} a_m .$$

Further, if $\{\lambda_n\}$ is any sequence of positive real numbers with $\lim_{n \to \infty} \lambda_n = 0$, then there exists a $\mu(\theta)$ satisfying (6.2.5) and (6.2.6) such that for the corresponding $f(z)$

$$a_m > \frac{B^m(0)}{m!} + K(m)\, \lambda_m$$

for an infinity of values of m, where $K(m)$ is the coefficient of $\sin^2 m\theta$ in $a_1(m)$ obtained by replacing $d\mu(\theta)$ by

$$d\mu(\theta) = \frac{1}{2} [1 + \sum_{k=3}^{\infty} \frac{\sin m_k \theta}{k^{s+1}}] \sin\theta \, d\theta$$

and $B^m(0)$ is the m^{th} derivative of $B(z) = \sum_{n=1}^{p-1} A_o^n \prod_{k=1}^{n} S(z, \cos\theta_k)$ at $z = 0$.

Proof : Replacing $S(z,\cos\phi)$ for $\phi = \theta, \theta_1, \theta_2, \ldots, \theta_k$ by $\sum_{n=1}^{\infty} \frac{\sin n\phi}{\sin\phi} z^n$ in (6.1.1), we obtain $f(z) = F(z) + G(z) \equiv \sum_{n=1}^{\infty} a_n z^n$, where,

$$F(z) = \sum_{m=1}^{p-1} A_o^m \prod_{k=1}^{m} \left(\sum_{m=1}^{\infty} \frac{\sin n\theta_k}{\sin\theta_k} z^n \right) \quad \text{and}$$

$$G(z) = \frac{1}{\pi} \prod_{k=1}^{p-1} \left(\sum_{n=1}^{\infty} \frac{\sin n\theta_k}{\sin\theta_k} z^n \right) \int_o^{\pi} \left(\sum_{n=1}^{\infty} \frac{\sin n\theta}{\sin\theta} z^n \right) d\mu(\theta)$$

From this, we get

$$(6.4.11) \qquad a_m = \frac{F^m(0)}{m!} + \int_o^{\pi} \sum_{i_{p-1}=p-1}^{m-1} \left[\sum_{i_{p-2}=p-2}^{i_{p-1}-1} \left[\sum_{i_{p-3}=p-3}^{i_{p-2}-1} \left[\ldots \left[\sum_{i_2=2}^{i_3-1} \left[\sum_{i_1=2}^{i_2} \frac{\sin i_1\theta_1 \sin (i_2-i_1)\theta_2}{\sin\theta_1 \sin\theta_2} \right] \frac{\sin (i_3-i_2)\theta_3}{\sin\theta_3} \right] \ldots \right] \frac{\sin(i_{p-1}-i_{p-2})\theta_{p-1}}{\sin\theta_{p-1}} \right] \frac{\sin (m-i_{p-1})\theta}{\sin\theta} \right] d\mu(\theta)$$

where $i_1, i_2, \ldots, i_{p-2}, i_{p-1}, m$ run over the set of natural numbers. Since $\mu(\theta)$ is absolutely continuous, $\mu'(\theta) \leq A\,\theta$ and $\mu'(\theta) \leq A\,(\pi-\theta)$ which together imply that $\frac{\mu'(\theta)}{\sin\theta}$ is an L-integrable function in the interval $[0,\pi]$. Thus, by Riemann-Lebesgue theorem [64], we have

$$\lim_{m\to\infty} a_m = \lim_{m\to\infty} \frac{F^m(0)}{m!} = \lim_{m\to\infty} \frac{B^m(0)}{m!}$$

For proving the second half of the theorem, we take any sequence $\{\lambda_m\}$ which

satisfies the hypotheses of theorem 5. Let $\{\lambda_{m_k}\}$ be a subsequence of $\{\lambda_m\}$ with the property that $\frac{\pi}{2k^{s+1}} > \lambda_{m_k}$ where $s > 0$ and m_k is an even intege

Define

$$(6.4.12) \qquad \mu'(\theta) = \frac{1}{2}\left[1 + \sum_{k=3}^{\infty} \frac{\sin m_k \theta}{k^{s+1}}\right] \sin\theta \,, \quad 0 < \theta < \pi.$$

Obviously $\mu(\theta)$ satisfies the conditions (6.2.5) and (6.2.6), and $f(z)$ as given by (6.1.1), with $\mu(\theta) = \int_0^\theta \mu'(\theta)\, d\theta$, defined by (6.4.12), belongs to the class $T(p)$. Moreover, if

$$f(z) = \sum_{m=1}^{\infty} a_m z^m$$

then a_m is given by (6.4.11) with $\mu(\theta)$ replaced by (6.4.12). It is clear from (6.4.11) that the contribution of the second part of (6.1.1) comes only in the case when $m = m_k$. If we denote this value by $K(m_k)$ then from (6.4.11) and (6.4.12) it follows that

$$a_{m_k} > \frac{F^{m_k}(0)}{m_k!} + \lambda_{m_k} K(m_k)$$

This completes the proof of theorem 5.

6.5. Remarks : By using Hardy-Stien-Spencer identity ([20], 47), it has been proved that if $f(z)$ belongs to the class $T(p)$ (or more generally to the class of mean-p-valent functions) and

$$(6.5.1) \qquad M(r,f) = O(1-r)^{-\lambda}$$

then

(6.5.2) $$L(r,f) = 0(1-r)^{-\lambda} \quad \text{for } \frac{1}{2} < \lambda \leq 2$$

By using theorem 3, we have obtained that

(6.5.3) $$L(r,f) = \begin{cases} 0(1-r)^{-\lambda-2(p-1)} & \text{if } 1 \leq \lambda \leq 2 \\ 0(1-r)^{-2p+1} & \text{if } 0 < \lambda < 1 \end{cases}$$

for $f(z) \varepsilon T^*(p)$ subject to the conditions

(6.5.4) $$f(r) = 0(1-r)^{-\lambda-2(p-1)} \quad \text{and} \quad f(-r) = 0(1-r)^{-\lambda-2(p-1)}$$

with $\theta_0 = 0$ and π respectively. It can be easily seen that (6.5.4) is a weaker assumption than that of (6.5.1). It is to be further noted that the class $T^*(p)$ forms an extremal class of typically real functions and therefore the results derived in corollaries 1 and 2, remain true for $f(z) \varepsilon T(p)$. Corollaries 1 and 2 also contain some of the results of Robertson [55] and A.W. Goodman and Robertson [18].

CHAPTER 7

Coefficient Estimates of Bounded Bazilevič p-valent Functions

7.1. Let

$$(7.1.1) \qquad f(z) = \{\beta \int_0^z h(\omega) g^{\beta}(\omega) d\omega\}^{1/\beta}$$

where $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ is regular in $|z| < 1$ and $\text{Re}\{h(z)\} > 0$, $g(z)$ is starlike regular in $|z| < 1$ with respect to the origin and $\beta > 0$. The functions represented by (7.1.1) can be equivalently characterized by the following condition :

$$(7.1.2) \qquad \text{Re}\{\frac{zf'(z)}{f^{1-\beta}(z)\ g^{\beta}(z)}\} \geq \rho > 0.$$

In each of the conditions (7.1.1) and (7.1.2) only principal values are assigned for $\beta > 0$. Bazilevic [2] introduced a more general class of functions $f(z)$ regular in $|z| < 1$ and defined by the following relation ;

$$(7.1.3) f(z)=\{\frac{\beta}{1+\alpha^2}\int_0^z (h(\omega)-\alpha i)\omega^{[-\alpha\beta i/1+\alpha^2]^{-1}}\{g(\omega)\}^{\beta/1+\alpha^2}\}^{\frac{1+\alpha i}{\beta}}$$

where $h(z)$, $g(z)$ and β are same as in (7.1.1), and α is any real number. He showed that any regular function $f(z)$ in $|z| < 1$, defined by (7.1.3) is univalent in $|z| < 1$. Thus , in particular, it follows that functions defined by (7.1.1) or (7.1.2) are also univalent in $|z| < 1$. Another proof of this result was recently given by Pommerenke [50]. In 1968, Thomas [66] identified the functions of (7.1.2) as Bazilevič regular functions of type β. In [66], Thomas introduced another class of mappings of $|z| > 1$ onto a domain containing ∞, that are analogous to the classes of Bazilevič functions of

type β. He defined this class as follows.

The function $f(z) = \sum_{n=0}^{\infty} a_n z^{-n}$ is called a meromorphic Bazilevic function of type β, if there exists a functinn $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$ which is starlike regular in $1 < |z| < \infty$ and satisfies (7.1.2) for $|z| > 1$ and β real.

A similar class of Bazilevic regular functions whose argument is of bounded variation has very recently been introduced by Nunokawa [40] and a coefficient problem is solved for this class.

In this chapter, we shall consider another analogous class of mappings f of $|z| < 1$ which are p-valent regular and (7.1.2) is satisfied for some p-valent starlike [56] regular function $g(z)$ in $|z| < 1$ and $\beta \geq 0$. We shall call such mappings p-valent Bazilevič functions.

7.2. In this section we shall state the following lemmas which we shall need in the next section.

Lemma 1([2],p.99). Let $f(z) = \sum_{n=0}^{\infty} a_{n+p} z^{n+p}$; $a_p = 1$ be mean p-valent function in $|z| < 1$. Then

$$\frac{r^p}{(1+r)^{2p}} \leq |f(z)| \leq \frac{r^p}{(1-r)^{2p}}$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(z)|^2 \, d\theta \leq \frac{p^2 (1+r)^2}{r^2 (1-r)^2} \cdot \frac{1}{2\pi} \int_0^{2\pi} |f(z)|^2 \, d\theta$$

Lemma 2. If $f(z) = \sum_{n=o}^{\infty} a_{n+p} z^{n+p}$; $a_p = 1$, be mean-p-valent in $|z| < 1$ and $f_n(z) = \int_o^z t^n f'(t)\, dt$, then

$$\frac{1}{2\pi}\int_o^{2\pi} |f_n(z)|^2\, d\theta \leq \frac{r^{2n}\, p^2(1+r)^2}{(p+n)^2(1-r)^2} \cdot \frac{1}{2\pi}\int_o^{2\pi}|f(z)|^2\, d\theta .$$

Proof. Let $f(z) = \sum_{n=o}^{\infty} a_{n+p} z^{n+p}$ with $a_p = 1$ be mean-p-valent analytic function in $|z| < 1$. Then

$$f_n(z) = \int_o^z t^n\left(\sum_{m=0}^{\infty} (p+m)a_{m+p}t^{m+p-1}\right)dt = \sum_{m=o}^{\infty} \frac{(p+m)a_{m+p}z^{n+m+p}}{n+m+p}$$

Hence,

$$\begin{aligned}\frac{1}{2\pi}\int_o^{2\pi} |f_n(z)|^2\, d\theta &= \sum_{m=o}^{\infty} \frac{(p+m)^2|a_{m+p}|^2\, r^{2(m+n+p)}}{(p+m+n)^2} \\ &\leq \frac{r^{2n+2}}{(p+n)^2} \sum_{m=o}^{\infty} (p+m)^2\, |a_{m+p}|^2\, r^{2(m-1)+2p} \\ &= \frac{r^{2n+2}}{(p+n)^2} \cdot \frac{1}{2\pi}\int_o^{2\pi} |f'(z)|^2\, d\theta\end{aligned}$$

and by lemma 1, we have

$$\frac{1}{2\pi}\int_o^{2\pi}|f_n(z)|^2\, d\theta \leq \frac{p^2(1+r)^2 r^{2n}}{(p+n)^2(1-r)^2}\, \frac{1}{2\pi}\int_o^{2\pi}|f(z)|^2\, d\theta$$

This completes the proof of lemma 2.

7.3. We shall prove the following :

Theorem. If $f(z) = \sum_{n=o}^{\infty} a_{n+p} z^{n+p}$, $a_p = 1$ be mean p-valent Bazilevic function in $|z| < 1$ and $|f(z)| \leq 1$, then

$$|a_n| \leq K(\beta,p)/n+\ell(\beta,p)$$

where $K(\beta,p)$ and $\ell(\beta,p)$ are absolute constants.

Proof. According to the hypothesis of the theorem we have

$$n|a_n| = \frac{1}{2\pi r^n} \int_0^{2\pi} f^{1-\beta}(z)\ g^{\beta}(z)\ \mathrm{Re}\{h(z)\}e^{-in\theta}\ d\theta\ -$$

$$- \frac{1}{2\pi r^n} \int_0^{2\pi} f^{1-\beta}(z)\ g^{\beta}(z)\ \bar{h}(z)e^{-in\theta}\ d\theta$$

$$\leq A(r) + B(r)$$

where,

$$A(r) = \left|\frac{1}{2\pi r^n} \int_0^{2\pi} f^{1-\beta}(z)\ g^{\beta}(z)\ \mathrm{Re}\{h(z)\}e^{-in\theta}\ d\theta\ \right|$$

$$\leq \frac{2\pi p K_1(\beta)}{\pi r^n}\ , \text{ where } K_1(\beta) = \max\ (1,(\frac{4}{r})^{p(\beta-1)})$$

and

$$B(r) = \left|\frac{1}{2\pi r^n} \int_0^{2\pi} f^{1-\beta}(z)\ g^{\beta}(z)\ \bar{h}(z)e^{-in\theta}\ d\theta\right|$$

$$\leq \frac{|\beta-1|}{\pi r^{2n}} \left|\int_0^{2\pi} f_n(z)\exp[2i(\beta-1)\arg\{f(z)\}-2i\beta\arg\{g(z)\}]\mathrm{Re}\{\frac{zf'(z)}{f(z)}\}d\theta\right|$$

$$+ \frac{\beta}{\pi r^{2n}}\left|\int_0^{2\pi} f_n(z)\exp[2i(\beta-1)\arg\{f(z)\}-2i\beta\arg\{g(z)\}]d_\theta(\arg\{g(z)\})\right|$$

$$\leq \frac{|\beta-1|}{\pi r^{2n}} \int_0^{2\pi}|f_n(z)|\left|\frac{zf'(z)}{f(z)}\right|d\theta + \frac{\beta}{\pi r^{2n}}\ .\ 2r^n\ .\ 2\pi p$$

$$\leq \frac{|\beta-1|}{\pi r^{2n}} \{\int_0^{2\pi} |f_n(z)|^2 d\theta \int_0^{2\pi} |\frac{zf'(z)}{f(z)}|^2 d\theta\}^{1/2} + \frac{4p\beta}{r^n}$$

From lemma 1 we have

$$\frac{1}{|f(z)|} \leq \frac{(1+r)^{2p}}{r^{2p}} \leq \frac{4^p}{r^{2p}}$$

Hence

$$B(r) \leq \frac{|\beta-1|2^p}{\pi r^{2n+p-1}} \{\int_0^{2\pi} |f_n(z)|^2 d\theta \int_0^{2\pi} |f'(z)|^2 d\theta\}^{\frac{1}{2}} + \frac{4\beta p}{r^n}$$

$$\leq \frac{|\beta-1|2^p\pi}{\pi r^{2n+p-1}} \{\frac{p^2(1+r)^2 r^{2n}.4\ p^2(1+r)^2}{(p+n)^2(1-r)^2 r^2 (1-r)^2}\}^{1/2} + \frac{4\beta p}{r^n}$$

$$= \frac{|\beta-1|2^{p+1}}{r^{n+p}} \frac{p^2(1+r)^2}{(p+n)(1-r)^2} + \frac{4\beta p}{r^n}$$

Hence we obtain

$$n|a_n| \leq \frac{4\beta p}{r^n} + \frac{|\beta-1|2^{p+1}}{r^{n+p}} \frac{p^2(1+r)^2}{(p+n)(1-r)^2} + \frac{2\pi p\ K_1(\beta)}{\pi r^n}$$

Let $r = 1 - \frac{1}{n}$, then

$$|a_n| \leq \{K_3(\beta,p) + \frac{K_2(\beta,p)}{n}\}$$

This completes the proof of the theorem.

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